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On Modified Burr XII-Inverse Exponential Distribution: Properties, Characterizations and Applications

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Abstract

In this paper, a flexible lifetime distribution with increasing, increasing and decreasing and modified bathtub hazard rate called Modified Burr XII-Inverse Exponential (MBXII-IE) is introduced. The density function of MBXII-IE has exponential, left-skewed, right-skewed and symmetrical shapes. Descriptive measures such as moments, moments of order statistics, incomplete moments, inequality measures, residual life function and reliability measures are theoretically established. The MBXII-IE distribution is characterized via different techniques. Parameters of MBXII-IE distribution are estimated using maximum likelihood method. The simulation study is performed to illustrate the performance of the Maximum Likelihood Estimates (MLEs) of the parameters of the MBXII-IE distribution. The potentiality of MBXII-IE distribution is demonstrated by its application to real data sets: fracture toughness, taxes revenue's data and coal mining disaster data

Keywords: Characterizations; Moments; Maximum Likelihood Estimation; Reliability

Introduction

A flexible model for the analysis of lifetime data sets is often attractive to the researchers. Inverse exponential is of interest due to its flexibility and simplicity.

In recent decades, many continuous univariate distributions have been developed but various data sets from reliability, insurance, finance, climatology, biomedical sciences and other areas do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to problems in these areas is a clear need of day.

The modified, extended and generalized distributions are obtained by the introduction of some transformation or addition of one or more parameters to the baseline distribution. These new developed distributions provide better fit to data than the sub and competing models.

Burr (1942) suggested twelve distributions as Burr family to fit cumulative frequency functions on frequency data. Burr distributions XII, III and X are frequently used. Burr-XII (BXII) distribution has a wide range of applications in modeling insurance data in finance and business and failure time data in reliability, survival and acceptance sampling plans.

Many modified, extended and generalized forms of BXII distribution are available in the literature such as BXII and related [1], doubly truncated Lomax [2], doubly truncated BXII [3], extended BXII [4], extended three-parameter BXII [5], six-parameter generalized BXII [6], beta BXII [7], extended BXII [4], Kumara swamy BXII [8], generalized log-Burr family [9], BXII geometric [10], McDonald BXII [11], three-Parameter BXII [12], BXII power series [13], three-parameter BXII Distribution [14], four-parameter BXII [15], BXII-Poisson [16], extensions of the BXII [17], new extended BXII[18], gamma BXII[19], BXII[20], BXII modified Weibull [21], five-parameter BXII [22], new BXII [23], Odd Lindley BXII [15,24] and modified log BXII distribution [25].

The main goal of this paper is to propose a more flexible distribution for the lifetime applications called modified. Burr XII - inverse exponential (MBXII-IE) distribution. The MBXII-IE density has exponential, left-skewed, right-skewed and symmetrical shapes. The MBXII-IE distribution has increasing, increasing and decreasing and modified bathtub hazard rate function. The flexible nature of the hazard rate function of MBXII-IE distribution helps to serve as a much better alternative model to the other current models for modelling real data in economics, life testing, reliability and survival analysis. The MBXII-IE distribution provides better fits than other sub-models.

Our interest is to study MBXII-IE distribution in terms of its mathematical properties, applications and comparison with the other sub-models.

This paper is sketched into the following sections. In Section 2, MBXII-IE distribution is introduced. In Section 3, some mathematical properties of MBXII-IE distribution are theoretically derived. In Section 4, stress-strength reliability and multi component stress-strength reliability of the model are studied. In Section 5, MBXII-IE distribution is characterized via (i) conditional expectation; (ii) truncated moment; (iii) hazard function; (iv) Mills ratio; (v) certain functions of the random variable; (vi)1^{st} order statistic and (vii) conditional expectation of the record values. In Section 6, the parameters of the MBXII-IE are estimated using maximum likelihood method. In Section 7, a simulation study is performed to illustrate the performance of the Maximum Likelihood Estimates (MLEs). In Section 8, the potentiality of the MBXII-IE distribution is demonstrated by its application to real data sets: river flows, taxes revenue's data and survival times. Goodness of fit of the probability distribution through different methods are studied. The concluding remarks are given in Section 9.

Development of MBXII-IE Distribution

The probability density function (pdf) and cumulative distribution function (cdf) of the inverse exponential distribution is given, respectively, by

$$g(x;\lambda) = \frac{\lambda}{x^2}\exp\left(-\frac{\lambda}{x}\right), \qquad x > 0, \lambda > 0,$$

and

$$G(x;\lambda) = \exp\left(-\frac{\lambda}{x}\right), x \ge 0, \lambda > 0$$

The odds ratio for the inverse exponential random variable X is

$$W\left(G(x)\right) = \frac{G(x;\lambda)}{\bar{G}(x;\lambda)} = \frac{\exp\left(-\frac{\lambda}{x}\right)}{1-\exp\left(-\frac{\lambda}{x}\right)} = \left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-1}$$

Gurvich et al. [26] replaced "x" with odds ratio in the Weibull distribution for the development of a class of extended Weibull distributions. Alzaatreh et al. [27] developed the cdf of the T-X family of distributions as

$$F\left(x\right) = \int_a^{W\left(G(x)\right)} r\left(t\right)dt$$

where W(G(x)) is function of G(x) and r(t) is the pdf of a non-negative random variable.

Bourguignon et al. [28] inserted the odds ratio of a baseline distribution in place of 'x' in the cdf of the Weibull distribution for the development of a new family of distributions.

The MBXII-IE is developed by inserting the odds ratio of the inverse exponential in place of 'x' in the cdf of MBXII distribution. The cdf for MBXII-IE distribution is obtained as

$$F(x) = \int_0^{W(G(x))} \alpha\beta t^{\beta-1}\left(1+\gamma t^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt$$

or

$$F(x;\alpha,\beta,\gamma,\lambda) = \int_0^{\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-1}} \alpha\beta t^{\beta-1}\left(1+\gamma t^{\beta}\right)^{-\frac{\alpha}{\gamma}-1}dt.$$

or

$$F(x) = 1-\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}, \alpha>0, \beta>0, \gamma>0, \lambda>0, x\ge 0. \quad (1)$$

The pdf of the MBXII-IE distribution is

$$f\left(x\right) = \alpha\beta\frac{\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}, \quad x>0. \quad (2)$$

The survival, hazard, reverse hazard, cumulative hazard functions and the Mills ratio of a random variable X with MBXII-IE distribution are given, respectively, by

$$S\left(x\right) = \left\{1+\gamma\left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}, \quad x\ge 0, \quad (3)$$

$$h\left(x\right) = \alpha\beta\frac{\lambda}{x^2}\exp\left(\frac{\lambda}{x}\right)\left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\left\{1+\gamma\left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-1}, \quad x>0, \quad (4)$$

$$r\left(x\right) = \frac{d}{dx}\ln\left(1-\left\{1+\gamma\left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}\right), \quad x>0, \quad (5)$$

$$H(x) = \frac{\alpha}{\gamma}\ln\left\{1+\gamma\left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}, \quad x>0, \quad (6)$$

and

$$m(x)=\frac{1}{\alpha\beta\lambda}x^{2}e^{-\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{\beta+1}\left[1+\gamma\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}\right]. \qquad (7)$$

The elasticity $e(x)=\frac{d\, lnF(x)}{D\, ln\, x}=xr(x)$ for the MBXII-IE distribution is

$$e(x)=\frac{d}{d\ln x}\ln\left(1-\left\{1+\gamma\left[\exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}\right). \qquad (8)$$

The elasticity of the MBXII-IE distribution shows the behaviour of the accumulation of probability in the domain of the random variable.

Mathematical Properties

Compounding and Transformations

The MBXII-IE distribution is derived through (i) ratio of exponential and gamma random variables and (ii) compounding generalized Weibull- inverse exponential (GW-IE) and gamma distributions.

Lemma (i) If the random variable Z_1 follows the exponential distribution with the parameter value 1 and the random variable Z_2 follows the fractional gamma i.e.

$Z_2 \sim$ Gamma $(\frac{\alpha}{\gamma},1)$,then for

$$Z_1=\gamma\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}Z_2.$$

we have

$$X=\ln\left[\left(\frac{\gamma Z_2}{Z_1}\right)^{\frac{1}{\beta}}+1\right]^{\lambda}\sim MBXII-IE(\alpha,\beta,\gamma,\lambda).$$

If Y| $\beta, \kappa, \gamma, \lambda, \theta \sim$ GW-IE (Y; $\beta, \kappa, \gamma, \lambda, \theta$) and $\theta|\alpha, \gamma \sim$ fractional gamma ($\theta; \alpha, \gamma$), then integrating the effect of θ with the help of $f(y, \alpha, \beta, \gamma, \lambda)=\int_0^{\infty} g(y, \alpha, \beta, \gamma, \lambda|\theta) g(\theta| \alpha, \gamma)d\theta$, we have

Y ~ MBXII - IE ($\alpha, \beta, \gamma, \lambda$).

Descriptive Measures Based on Quantiles

The quantile function of MBXII-IE distribution is

$$x_q=\lambda\left(\ln\left\{\left[\frac{(1-q)^{-\frac{\gamma}{\alpha}}-1}{\gamma}\right]^{-\frac{1}{\beta}}+1\right\}\right)^{-1}.$$

The median of MBXII-IE distribution is

$$Median=\lambda\left\{\ln\left[\left(\frac{\gamma}{2^{\frac{\gamma}{\alpha}}-1}\right)^{\frac{1}{\beta}}+1\right]\right\}^{-1}$$

MBXII-IE random number generator is

$$X=\lambda\left(\ln\left\{\left[\frac{(1-Z)^{-\frac{\gamma}{\alpha}}-1}{\gamma}\right]^{-\frac{1}{\beta}}+1\right\}\right)^{-1},$$

where the random variable Z has the uniform distribution on (0,1). Some measures based on quartiles for location, dispersion, skewness and kurtosis for the MBXII-IE distribution respectively are: Median M=$Q_{1}$and Quartile deviation is

$$Q.D.=\frac{Q_{\frac{3}{4}}-Q_{\frac{1}{4}}}{2}$$

Bowley's skewness measure is

$$S_q=\frac{Q_{\frac{3}{4}}-2Q_{\frac{1}{2}}+Q_{\frac{1}{4}}}{Q_{\frac{3}{4}}-Q_{\frac{1}{4}}}$$

and Moors kurtosis measure based on Octiles is

$$K=\frac{Q_{\frac{7}{8}}-Q_{\frac{5}{8}}+Q_{\frac{3}{8}}-Q_{\frac{1}{8}}}{Q_{\frac{6}{8}}-Q_{\frac{2}{8}}}$$

The quantile based measures exist even for the models that have no moments. The quantile based measures are less sensitive to the outliers.

Sub-Models

The MBXII-IE distribution has the following sub models shown in Table 1.

1	α	β	γ	λ	MBXII-IE distribution
2	α	β	1	λ	BXII-IE distribution
3	α	1	1	λ	Lomax-IE distribution
4	1	β	1	λ	Log-logistic-IE distribution
5	α	β	$\gamma\to 0$	λ	Weibull-IE distribution

Table 1: Sub-models of the MBXII-IE Distribution.

Shapes of the MBXII-IE Density and Hazard Rate Functions

The following graphs show that shapes of the MBXII-IE density are exponential, positively, negatively skewed and symmetrical Figure 1. The MBXII-IE distribution has increasing, increasing and decreasing and modified bath tub hazard rate function Figure 2.

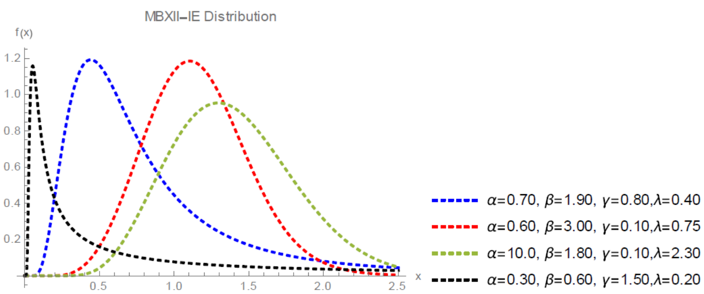

Figure 1: Plots of pdf of the MBXII-IE distribution for selected parameter values.

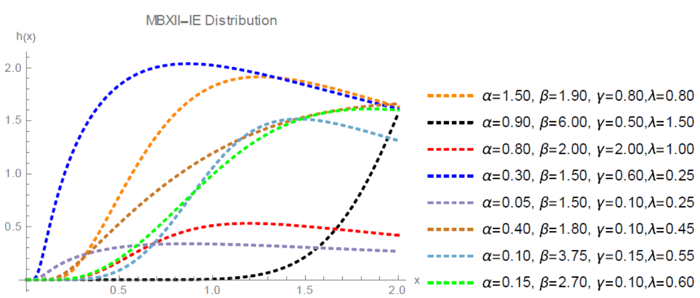

Figure 2: Plots of hrf of the MBXII-IE distribution for selected parameter values.

Linear Representation

Hereafter, we denote by X ~ MBXII - IE (x, α, β, γ, λ) a Random Variable (r. v) having density function (2). In this section, we provide a linear representation for the density of X to derive some mathematical quantities for the MBXII-IE model. The cdf of X can be expressed as

$$F(x)=1-\underbrace{\left(1+\gamma\left\{\frac{\exp\left[-\left(\frac{\lambda}{x}\right)\right]}{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]}\right\}^{\beta}\right)^{-\frac{\alpha}{\gamma}}}_{A}. \quad (9)$$

First, we will consider the following two power series

$$(1+s)^{-c}=\sum_{k=0}^{\infty}2^{-c-k}(s-1)^{k}\binom{-c}{k}, \quad (10)$$

and

$$(1-s)^{-c}=\sum_{j=0}^{\infty}\frac{\Gamma(c+j)}{j!\Gamma(c)}s^{j}\,\big|_{(|s|<1,c>0)}. \quad (11)$$

Applying (10) for A in (9) gives

$$F(x)=1-\sum_{k=0}^{\infty}2^{-\frac{\alpha}{\gamma}-k}\binom{-\frac{\alpha}{\gamma}}{k}\left(\gamma\left\{\frac{\exp\left[-\left(\frac{\lambda}{x}\right)\right]}{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]}\right\}^{\beta}-1\right)^{k}$$

Second, using the binomial expansion, the last equation can be expressed as

$$F(x)=1-\sum_{k=0}^{\infty}\sum_{i=0}^{k}\left(\begin{array}{c}\frac{(-1)^{i}\gamma^{k-i}}{2^{\frac{\alpha}{\gamma}+k}}\binom{k}{i}\binom{-\frac{\alpha}{\gamma}}{k}\\ \times\left\{\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{(k-i)\beta}\underbrace{\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{-(k-i)\beta}}_{B}\end{array}\right)$$

Third, applying (11) for B in the last equation gives

$$F(x)=1-\sum_{j,k=0}^{\infty}\sum_{i=0}^{k}d_{i,j,k}\left\{\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{(k-i)\beta+j},$$

$$F(x)=1-\sum_{j,k=0}^{\infty}\sum_{i=0}^{k}d_{i,j,k}H_{(k-i)\beta+j}(x), \quad (12)$$

where h_c (x) is the cdf of the IE model with scale parameter c and

$$d_{i,j,k}=\frac{(-1)^{i}\gamma^{k-i}\Gamma([k-i]\beta+j)}{2^{\frac{\alpha}{\gamma}+k}j!\Gamma(\Gamma([k-i]\beta))}\binom{k}{i}\binom{-\frac{\alpha}{\gamma}}{k}$$

Upon differentiating (12), we obtain

$$f(x)=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}h_{(k-i)\beta+j}(x), \quad (13)$$

where $h_c(x)$ denotes the pdf of the IE model with scale parameter and

$$\tau_{i,j,k}=-d_{i,j,k}$$

Equation (13) is the main result of this section. It reveals that the MBXII-IE density is a linear combination of IE densities. So, some of its mathematical properties can be easily determined from those of IE model.

Moments

The r^{th} ordinary moment of X μ'_r = E (X^r), is determined from (13) as

$$\mu'_r=E\left(X^r\right)=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}\left\{\lambda\left[(k-i)\beta+j\right]\right\}^r\Gamma(1-r)\Big|_{(r<1)}$$

The k^{th} moment about mean of X is determined from the relationship

$$\mu_k=E\left[X-E(X)\right]^k=\sum_{j=1}^{k}\binom{k}{j}(-1)^j\mu'_j\mu'_{(k-j)}$$

The Pearson's measure of skewness γ_1, Kurtosis β_2, moment generating function and cumulants can be calculated from

$$\gamma_1=\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},\ \beta_2=\frac{\mu_4}{(\mu_2)^2},M_X(t)=\sum_{r=1}^{\infty}\frac{t^r}{r!}E(X)^r \text{ and } k_r=E\left(X^r\right)-\sum_{c=1}^{r-1}\binom{r-1}{c-1}k_cE\left(X^{r-c}\right)$$

The r^{th} incomplete moment of X, say $\varphi_r(t)$, can be determined from (13) as

$$\varphi_r(t)=\int_{-\infty}^{t}x^rf(x)dx$$

$$=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}\left\{\lambda\left[(k-i)\beta+j\right]\right\}^r\left[\gamma\left(1-r,\left(\frac{\lambda}{t}\right)\right)\right]\Big|_{(r<1)}$$

Where $\gamma(\varsigma, q)$ is the incomplete gamma function.

$$\gamma(a,q)\Big|(a\neq 0,-1,-2,.....)=\int_{0}^{q}t^{a-1}\exp(-t)dt$$

$$=\frac{q^a}{a}\left\{{}_1F_1\left[a;a+1;-q\right]\right\}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(a+k)}q^{a+k}$$

and ${}_1F_1$.,.,. is a confluent hyper geometric function? The moment generating function (mgf) M(t) = E(e^{tx}) of

X follows from (13) as

$$M(t)=\sum_{(j,k=0|j+k\geq 1)}^{\infty}\sum_{r=0}^{\infty}\sum_{i=0}^{k}\left(t^r\tau_{i,j,k}/r!\right)\left\{\lambda\left[(k-i)\beta+j\right]\right\}^r\Gamma(1-r)\Big|_{(r<1)}$$

Probability Weighted Moments (PWMs)

The (s, r)th PWM of X ,denoted by $\rho_{s,r}$, is formally defined by

$$\rho_{s,r}=E\left\{X^sF(X)^r\right\}=\int_{-\infty}^{\infty}x^sF(x)^rf(x)dx$$

Using (1), we have

$$F(x)^r=\left[1-\left(1+\gamma\left\{\frac{\exp\left[-\left(\frac{\lambda}{x}\right)\right]}{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]}\right\}^{\beta}\right)^{-\frac{\alpha}{\gamma}}\right]^r$$

Expanding z^λ in Taylor series, we can write

$$s^\lambda=\sum_{i=0}^{\infty}\frac{(\lambda)_i}{i!}(s-1)^i=\sum_{i=0}^{\infty}f_i(\lambda)s^i, \quad (14)$$

where $(\lambda)_i = \lambda(\lambda-1)\ldots(\lambda-i+1)$ is the descending factorial and

$$f_i(\lambda)=\sum_{i=0}^{\infty}\left\{\left[(-1)^{h-1}(\lambda)_h\right]/h!\right\}\binom{h}{i}$$

First, applying the Taylor series in S^λ for F $(x)^r$ we obtain

$$F(x)^r=\sum_{i=0}^{\infty}(-1)^if_i(r)\left\{1+\gamma\left[\frac{\exp\left[-\left(\frac{\lambda}{x}\right)\right]}{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]}\right]^{\beta}\right\}^{-\frac{\alpha i}{\gamma}}$$

Second, using (2) and the last equation, we have

$$f(x)^rF(x)^r=\beta\alpha\gamma^{-1}\lambda x^{-2}\exp\left[-\left(\frac{\lambda}{x}\right)\right]\frac{\left\{\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{\beta-1}}{\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{\beta+1}}\sum_{i=0}^{\infty}(-1)^if_i(r)$$

$$\times\underbrace{\left\{1+\gamma\left[\frac{\exp\left[-\left(\frac{\lambda}{x}\right)\right]}{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]}\right]^{\beta}\right\}^{-(i+1)\frac{\alpha}{\gamma}-1}}_{C}$$

Applying (10) for C in the last equation, we obtain

$$f(x)F(x)^r=\beta\alpha\gamma^{-1}\sum_{i,k=0}^{\infty}(-1)^if_i(r)\lambda x^{-2}\exp\left[-\left(\frac{\lambda}{x}\right)\right]\frac{\left\{\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{\beta-1}}{\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{\beta+1}}2^{-(i+1)\frac{\alpha}{\gamma}-k-1}$$

$$\times\binom{-(i+k)\frac{\alpha}{\gamma}-1}{k}\underbrace{\left(1+\gamma\left\{\frac{\exp\left[-\left(\frac{\lambda}{x}\right)\right]}{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]}\right\}^{\beta}\right)^{k}}_{D}$$

Third, using the binomial expansion for D, the last equation can be rewritten as

$$f(x)F(x)^{r}=\gamma^{k-j-i}\beta\alpha\lambda x^{-2}\exp\left[-\left(\frac{\lambda}{x}\right)\right]\sum_{i,k=0}^{\infty}\sum_{j=0}^{k}(-1)^{i+j}\binom{k}{j}\binom{-(i+1)\frac{\alpha}{\gamma}-k-1}{k}$$

$$\times f_{i}(r)\left\{\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{(k-j+1)\beta-1}2^{-(i+1)\frac{\alpha}{\gamma}-k-1}\underbrace{\left\{1-\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{-[(k-j+1)\beta+1]}}_{E}$$

Applying (11) for E in the last equation gives

$$f(x)F(x)^{r}=\sum_{i,k,m=0}^{\infty}\sum_{j=0}^{k}\gamma^{k-j-1}\beta\alpha\frac{(-1)^{i+j}f_{i}(r)\Gamma\left([k-j+1]\beta+m+1\right)}{2^{(i+1)\left(\frac{\alpha}{\gamma}\right)+k+1}m!\Gamma\left([k-j+1]\beta+1\right)}$$

$$\times\frac{\left[(k-j+1)\beta+m+1\right]}{\left[(k-j+1)\beta+m+1\right]}$$

$$\times\binom{k}{j}\binom{-\left[(i+1)\frac{\alpha}{\gamma}+1\right]}{k}\lambda x^{-2}\left\{\exp\left[-\left(\frac{\lambda}{x}\right)\right]\right\}^{(k-j+1)\beta+m}$$

and then

$$f(x)F(x)^{r}=\sum_{k,m=0}^{\infty}\sum_{j=0}^{k}a_{j,k,m}^{(r)}h_{(k-j+1)\beta+m+1}(x)$$

where

$$a_{j,k,m}^{(r)}=\gamma^{k-j-1}\beta\alpha\,\upsilon_{j,k,m}f_{i}(r),\qquad(15)$$

and $f_i(r)$is defined in (14), then (for j ≤ k)

$$u_{j,k,m}=\sum_{i=0}^{\infty}\frac{(-1)^{i+j}\left[\left([k-j+1]\beta+m+1\right)^{(m)}\right]}{2^{(i+1)\left(\frac{\alpha}{\gamma}\right)+k+1}\left[(k-j+1)\beta+m+1\right]m!}\binom{k}{j}\binom{-(i+1)\frac{\alpha}{\gamma}-k-1}{k}$$

where,

$$a^{(n)}=\Gamma(a+n)/\Gamma(a)$$

denotes the rising factorial. Finally, the (s, r)th PWM of X can be determined by

$$\rho_{s,r}=\sum_{k,m=0}^{\infty}\sum_{j=0}^{k}a_{j,k,m}^{(r)}\left\{\lambda\left[(k-j+1)\beta+m+1\right]\right\}^{s}\Gamma(1-s)\Big|_{(s<1)}$$

Residual Life and Reversed Residual Life Functions

The nth moment of the residual life, say $m_n(t)$, is

$$m_{n}(t)=E\left[\left(X-t\right)^{n}\Big|X>t\right],n=1,2,,...$$

which uniquely determines F(x). The nth moment of the residual life of X is given by

$$m_{n}(t)=\frac{\int_{t}^{\infty}(x-t)^{n}\,dF(x)}{1-F(t)}$$

Therefore

$$m_{n}(t)=\frac{1}{1-F(t)}\sum_{(j,k=0|j+k\geq1)}^{\infty}\sum_{i=0}^{k}\tau_{i,j,k}^{*}\left\{\lambda\left[(k-i)\beta+j\right]\right\}^{r}\Gamma\left(1-r,\left(\frac{\lambda}{t}\right)\right)\Big|_{(r<1)}$$

where

$$\Gamma(a,q)\Big|_{(q>0)}=\int_{q}^{\infty}t^{a-1}\exp(-t)dt$$

$$\Gamma(a,q)+\gamma(a,q)=\Gamma(a).$$

and

$$\tau_{i,j,k}^{*}=\tau_{i,j,k}(1-t)^{n}$$

The Mean Residual Life (MRL) or the life expectation at age t defined by

$$m_{1}(\mathrm{t})=E\left[\left(X-t\right)\Big|X>t\right],n=1$$

which represents the expected additional life length for a unit which is alive at age t. The MRL of X can be obtained by setting n=1 in the last equation.

The nth moment of the reversed residual life, say $M_n(t)$, is

$$M_{n}(t)=E\left[\left(X-t\right)^{n}\Big|X\geq t\right],\forall t>0,n=1,2,,...,$$

which uniquely determines F(x). We obtain

$$M_{n}(t)=\frac{\int_{0}^{t}(t-x)^{n}\,dF(x)}{F(t)}$$

Then, the nth moment of the reversed residual life of X becomes

$$M_n(t) = \sum_{(j,k=0|j+k\geq 1)}^{\infty} \sum_{i=0}^{k} \tau_{i,j,k}^{**} \left\{\lambda\left[(k-i)\beta + j\right]\right\}^r \left[\gamma\left(1-r,\left(\frac{\lambda}{t}\right)\right)\right]\Big|_{(r<1)}$$

Where

$$\tau_{i,j,k}^{**} = \tau_{i,j,k} \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{n-r}$$

The mean inactivity time (MIT), also called the mean reversed residual life function, is given by

$$M_1(t) = E\left[\left(X-t\right)\middle| X \geq t\right], \forall t > 0, n = 1$$

and it represents the waiting time elapsed since the failure of an item on condition that this failure had occurred in (0, t). The MIT of the MBXII-IE model is obtained easily by setting n=1 in the above equation.

Order Statistics

Order statistics have many applications in different areas of statistical theory and practice. Let $X_1,..., X_n$ be a random sample from the MBXII-IE model and let $X_{1:n},...,X_{n:n}$ be the corresponding order statistics. The pdf of the ith order statistic, $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{f(x)}{B(i, n-i+1)} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} F^{r+i-1}(x)$$

where B (.,.) is the beta function. From (15)

$$f(x)F(x)^{r+1-1} = \sum_{k,m=0}^{\infty} \sum_{j=0}^{k} a_{j,k,m}^{(r+i-1)} h_{(k-j+1)\beta+|m+1}(x)$$

Where $a_{j,k,m}^{(r+i+1)}$ is defined there. So, the pdf of $X_{i:n}$ is

$$f_{i:n}(x) = \frac{1}{\mathrm{B}(i, n-i+1)} \sum_{k,m=0}^{\infty} \sum_{r=0}^{n-i} \sum_{j=0}^{k} (-1)^r \binom{n-i}{r} a_{j,k,m}^{(r+i-1)}\; h_{(k-j+1)\beta+m+1}(x). \quad (16)$$

The qth ordinary moment of $X_{i:n}$, say $u\frac{r}{q}$ = E $(X\frac{q}{i:n})$, , is determined from (16) as

$$E(X_{i:n}^q) = \frac{1}{\mathrm{B}(i, n-i+1)} \sum_{k,m=0}^{\infty} \sum_{r=0}^{n-i} \sum_{j=0}^{k} (-1)^r \binom{n-i}{r} a_{j,k,m}^{(r+i-1)}$$

$$\times \left\{\lambda\left[(k-j+1)\beta + m + 1\right]\right\}^q \Gamma\left(1-q\right)|_{(q<1)}$$

Reliability Measures

In this section, different reliability measures for the MBXII-IE distribution are studied.

Stress-Strength Reliability

Let X_1~ MBXII –IE (α_1, β, γ, λ), X_2~ MBXII –IE (α_2, β, γ, λ) and let X_1 strength and X_2 represent stress. Then the reliability of the component is:

$$\mathrm{R} = \Pr\left(\mathrm{X}_2 < X_1\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{x_1} f\left(x_1, x_2\right) dx_2 dx_1 = \int_0^{\infty} f_{x_1}(x) F_{x_2}(x) dx$$

$$R = \int_0^{\infty} \frac{\alpha_1\beta\lambda}{x^2} exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}-1}\left(1-\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha_2}{\gamma}}\right)dx,$$

$$R = \frac{\alpha_2}{(\alpha_1 + \alpha_2)}. \quad (17)$$

(i) R is independent of β, γ and λ

(ii) For $\alpha_1 = \alpha_2$, R=0.5, X_1 and X_2 are independently and identically distrubuted, i.e. R = Pr (X_2<X_1) = Pr (X_1<X_2).

Multi Component Stress-Strength Reliability Estimator $R_{s,k}$

Suppose a machine has at least "s" components working out of "κ" components. The strengths of all components of the system are X_1, X_2,...,X_κ and stress Y is applied to the system. Both the strengths 1 X_1, X_2,...,X_κ are i.i.d. and are independent of stress Y. The cdf of Y is G and F is the cdf of X. The reliability of a machine is the probability that the machine functions properly.

Let X~ MBXII-IE (α_1, β, γ, λ), Y~ MBXII-IE (α_2, β, γ, λ) with common parameters β, γ, λ and unknown shape parameters α_1 and α_2. The multi component stress- strength reliability for MBXII-IE distribution is given by

$$R_{s,k} = P\left(strengths > stress\right) = P\left[at\,least\,"s"\,of\,\left(X_1, X_2, \ldots X_m\right) exceed\, Y\right], \quad (18)$$

$$R_{s,\kappa} = \sum_{l=s}^{\kappa}\binom{k}{\ell}\int_0^{\infty}[1-F(y)]^l\,[F(y)]^{\kappa-l}\,dG(y), \quad \text{(Bhattacharyya and Johnson; 1974).} \quad (19)$$

$$R_{s,\kappa} = \sum_{l=s}^{\kappa}\binom{\kappa}{l}\int_0^{\lambda}\left(\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}}\right)^l\left(1-\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha_1}{\gamma}}\right)^{(\kappa-l)}$$

$$\times\frac{\alpha_2\beta\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha_2}{\gamma}-1}dx$$

Let

$t=\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha_2}{\gamma}}$, then we obtain

$$R_{s,\kappa}=\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1\left(t^{v}\right)^{\ell}\left(1-t^{v}\right)^{(\kappa-\ell)}dt$$

Let

$z=t^{\nu}, t=z^{\frac{1}{\nu}}, \ dt=\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz$, then

$$R_{s,\kappa}=\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1(z)^{\ell}(1-z)^{(\kappa-\ell)}\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz$$

$$R_{s,\kappa}=\frac{1}{\nu}\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}\int_0^1(z)^{\ell+\frac{1}{\nu}-1}(1-z)^{(\kappa-\ell)}dz$$

$$R_{s,\kappa}=\frac{1}{\nu}\sum_{\ell=s}^{\kappa}\binom{\kappa}{\ell}B\left(\ell+\frac{1}{\nu},\kappa-\ell+1\right), where\ v=\frac{\alpha_1}{\alpha_2}. \qquad (20)$$

The probability $R_{s,k}$ in (20) is called reliability in a multi component stress-strength model.

Characterizations

In this section, the MBXII-IE distribution is characterized via: (i) conditional expectation; (ii) truncated moment; (iii) hazard function; (iv) Mills ratio; (v) certain functions of the random variable; (vi)1st order statistic and (vii) conditional expectation of the record values.

We present our characterizations in seven subsections.

Characterization via Conditional Expectation

The MBXII-IE distribution is characterized via conditional expectation.

Proposition 7.1.1: Let X: $\Omega \to (0, \infty)$ be a continuous random variable with cdf F (x) (0 < F (x) < 1 for x>0), then for $\alpha > \gamma$, X has cdf (1) if and only if

$$E\left\{\left[exp\left(\frac{\lambda}{X}\right)-1\right]^{-\beta}\middle|X>t\right\}=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\right\}\quad for\ t>0. \qquad (21)$$

Proof: If X has cdf (1), then

$$E\left\{\left[exp\left(\frac{\lambda}{X}\right)-1\right]^{-\beta}\middle|X>t\right\}=\left[1-F(t)\right]^{-1}\int_t^{\lambda}\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}f(x)dx$$

$$=\left[1-F(t)\right]^{-1}\int_t^{\lambda}\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\frac{\alpha\beta\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}$$

$$\times\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}-1}dx$$

Upon integration by parts and simplification, we arrive at

$$E\left\{\left[exp\left(\frac{\lambda}{X}\right)-1\right]^{-\beta}\middle|X>t\right\}=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\right\}\quad for\ t>0$$

Conversely, if (21) holds, then

$$\frac{1}{\overline{F}(t)}\int_t^{\infty}\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}f(x)dx=\frac{1}{(\alpha-\gamma)}\left\{1+\alpha\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\right\}$$

$$\frac{1}{\overline{F}(t)}\int_t^{\infty}\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}f(x)dx=\left(\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}+\frac{1}{(\alpha-\gamma)}\left\{1+\gamma\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\right\}\right). \qquad (22)$$

Differentiating (22) with respect to t, we obtain

$$-\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}f(t)=\beta\lambda e^{\frac{\lambda}{t}}\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta-1}t^{-2}\overline{F}(t)\left[1+\left(\frac{\alpha}{\gamma}-1\right)^{-1}\right]$$

$$-f(t)\left(\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}+\frac{1}{(\alpha-\gamma)}\left\{1+\gamma\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\right\}\right)$$

After simplification and integration, we arrive at

$$F(t)=1-\left\{1+\gamma\left[exp\left(\frac{\lambda}{t}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}, t\geq 0.$$

Characterizations via Truncated Moment of a Function of the Random Variable

Here we characterize the MBXII-IE distribution via relationship between truncated moment of a function of X and another function. This characterization is stable in the sense of weak convergence [29].

Proposition 7.2.1: Let X: $\Omega \to (0, \infty)$ be a continuous random variable and let

$$g(x)=\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-1}\quad for\ x>0$$

For x > 0. The pdf of X is (2) if and only if the function h(x), in Theorem 1 (Appendix A), has the form h (x)

$$= \frac{\alpha}{\alpha+\gamma}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-1}, \quad x>0$$

Proof: If X has pdf (2), then

$$\left(1-F(x)\right)E\left(g(X)\middle|X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\left(\frac{\alpha}{\gamma}+1\right)}, x>0$$

or

$$E\left(g(X)\middle|X\geq x\right)=\frac{\alpha}{\alpha+\gamma}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-1}, \quad x>0$$

and

Conversely, if h(x) is given as above, then

$$h'(x)=-\frac{\alpha}{\alpha+\gamma}\frac{\beta\gamma\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-2}<0, \text{ for } x>0.$$

and

$$s'(x)=\frac{h'(x)}{h(x)-g(x)}=\frac{\frac{\alpha}{\gamma}\frac{\beta\gamma\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}}{\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}}, \quad x>0$$

And hence

$$s(x)=\ln\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{\frac{\alpha}{\gamma}}, x>0.$$

and

$$e^{-s(x)}=\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha}{\gamma}}, x>0$$

Now, in view of Theorem 1, X has density (2).

Corollary 7.2.1: Let X : $\Omega\rightarrow$ (0, ∞) be a continuous random variable. The Pdf of X is (2) if and only if there

exist functions h(x) and g(x) defined in Theorem 1satisfying the differential equation

$$\frac{h'(x)}{h(x)-g(x)}=\frac{\frac{\alpha}{\gamma}\frac{\beta\gamma\lambda}{x^2}e^{\frac{\lambda}{x}}\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta-1}}{\left[1+\gamma\left(e^{\frac{\lambda}{x}}-1\right)^{-\beta}\right]}, \quad x>0$$

Remark 7.2.1: The general solution of the differential equation in Corollary 5.2.1 is

$$h(x)=\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{\frac{\alpha}{\gamma}}\left[-\int\frac{\frac{\alpha}{\gamma}\frac{\beta\gamma\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}}{\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{\frac{\alpha}{\gamma}+1}}g(x)dx+D\right]$$

where D is a constant.

Characterization via Hazard Function

In this sub-section, the MBXII-IE distribution is characterized in terms of the hazard function.

Definition 7.3.1: Let X be a continuous random variable with Pdf f(x) and cdf F(x). The hazard function $h_F(x)$ of a twice differentiable distribution function F(x) satisfies the differential equation

$$\frac{d}{dx}\left[\ln f(x)\right]=\frac{h_F'(x)}{h_F(x)}-h_F(x)$$

Proposition 7.3.1: Let X: Ω (0→∞) be continuous random variable. The pdf of X is (2) if and only if its hazard function, h_F (x), satisfies the first order differential equation

$$\left\{\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{\beta+2}+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{2}\right\}x^3e^{-\frac{\lambda}{x}}h_F'(x)+\beta\lambda x\left[exp\left(\frac{\lambda}{x}\right)-1\right]h_F(x)$$

$$=\alpha\beta\lambda\left\{2\left[exp\left(\frac{\lambda}{x}\right)-1\right]+\frac{\lambda}{x}\left(1+\beta exp\left(\frac{\lambda}{x}\right)\right)\right\}$$

Proof: If X has pdf (2), then the above differential equation holds. Now if the differential equation holds, then

$$\frac{d}{dx}\left(\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}h_f(x)\right)=\alpha\beta\lambda\frac{d}{dx}\left\{x^{-2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\right\}$$

or

$$h(x)=\alpha\beta\frac{\lambda}{x^2}exp\left(\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta-1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-1}, \quad x>0.$$

which is the hazard function of the MBXII-IE distribution.

Characterization via Mills Ratio

In this sub-section, the MBXII-IE distribution is characterized via Mills ratio.

Definition 7.4.1: Let X: Ω (0→∞) be a continuous random variable with cdf F(x) and pdf f(x). The Mills ratio, m(x), of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{d}{dx}\left[\ln f(x)\right]=-\left[\frac{1}{m(x)}+\frac{m'(x)}{m(x)}\right]$$

Proposition 7.4.1: Let X: Ω (0→∞) be continuous random variable. The pdf of X is (2) if and only if its Mills ratio satisfies the first order differential equation

$$m_F'(x)-\frac{\lambda}{x^2}m_F(x)=\frac{1}{\alpha\beta\lambda}\left\{2x\left(1-e^{-\frac{\lambda}{x}}\right)\left[\gamma+\left(e^{\frac{\lambda}{x}}-1\right)^{\beta}\right]-\lambda(\beta+1)\left[\gamma+\left(e^{\frac{\lambda}{x}}-1\right)^{\beta}\right]+\beta\gamma\lambda\right\}$$

Proof: If X has pdf (2), then the above differential equation surely holds. Now if the differential equation holds, then

$$\frac{d}{dx}\left[m_F(x)\,exp\left(-\frac{\lambda}{x}\right)\right]=\frac{d}{dx}\left(\frac{1}{\alpha\beta\lambda}x^2\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{\beta+1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}\right)$$

or

$$m(x)=\frac{1}{\alpha\beta\lambda}x^2exp\left(-\frac{\lambda}{x}\right)\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{\beta+1}\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}$$

which is Mills ratio of the MBXII-IE distribution.

Characterization via Certain Function of the Random Variable

The MBXII-IE distribution is characterized through certain function of the continuous random variable X. Used this technique for characterization.

Proposition 7.5.1: Let X: Ω (0→∞) be continuous random variable with cdf F(x) and pdf f(x). Let ψ(x) and φ(x) be differentiable functions on (0, ∞) such that

$$\int_0^{\infty}\frac{\psi'(x)}{\psi(x)-\varphi(x)}dx=\infty$$

Then E (φ(X)|X> x) = ψ (x), x > 0, implies F (x) = 1- exp

$$F(x)=1-\exp\left[-\int_0^x\frac{\psi'(t)}{\psi(t)-\varphi(t)}dt\right],\ x\geq 0.$$

Proof: We have

$$\int_x^{\infty}\varphi(u)f(u)du=\left(1-F(x)\right)\psi(x)$$

After differentiation of the above equation with respect to x, and then reorganizing the terms, we obtain

$$\frac{f(x)}{1-F(x)}=\frac{\psi'(x)}{\psi(x)-\varphi(x)},\ \ x>0$$

Integrating the last equation from 0 to x, we have

$$F(x)=1-\exp\left[-\int_0^x\frac{\psi'(t)}{\psi(t)-\varphi(t)}dt\right],\ x\geq 0$$

Remark 7.5.1: Taking

$$\varphi(x)=\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{\frac{\alpha}{2\gamma}}$$

And ψ(x)=2φ(x), Proposition **7.5.1** provides a characterization of (1). Clearly there are other choices of these functions.

Characterization Through Distribution of the 1st Order Statistic

Here we characterize the MBXII-IE distribution via distribution of the1st order statistic.

Proposition 7.6.1: Let X_1 X_2,..., X_n be i.i.d. random variables. If the distribution of X_i' S is MBXII-IE, then$X_{1:n}$ has the MBXII-IE distribution.

Proof: Observe that

$$P_r\left(X_{1:n}>x\right)=\left[P_r\left(X_1>x\right)\right]^n=\left[1-F(x)\right]^n$$

$$P_r\left(X_{1:n}>x\right)=\left\{1+\gamma\left[exp\left(\frac{\lambda}{x}\right)-1\right]^{-\beta}\right\}^{-\frac{\alpha n}{\gamma}}$$

Characterization via Conditional Expectation of the Record Values

Nagarajah H (1988) [31], Arnold BC (1988) et al. [32], Khan AH et al. (2004) [33], Athar H et al. (2014) [34] characterized distributions via conditional expectation of the record values.

Proposition 7.7.1: Let X: Ω (0→∞) be a continuous random variable with cdf F(x) and pdf f(x). Let $X_{U(r)}$ be the r^{th} record value of a random sample $X_1, X_2, \ldots X_n$. Then for two successive values $X_{U(r)}$ and $X_{U(s)}$,

$$1 \le r < s \le n,\ E[(h(X_{U(s)} - X_{U(r)}))^q \mid X_{U(r)} = x] = a^* \sum_{j=0}^{q} \binom{q}{j} (h(x))^{q-j} (b/a)^j$$

Holds, if and only if

$$F(x) = 1 - [a + bh(x)]^c,\ a \neq 0, \text{ where } a^* = \sum_{i=0}^{q} \binom{q}{i} (-1)^{i+q} \left(\frac{c}{c+i}\right)^{s-r} \text{ and } h(x)$$

and h(x) is a differentiable function of x.

Remark 7.7.1: Taking a=1, b=γ, $h(x) = [\exp(\frac{\lambda}{x}) - 1]^{-\beta}$, $c = -\frac{\alpha}{\gamma}$,

Proposition 7.7.1: provides a characterization of MBXII-IE distribution.

Maximum Likelihood Estimation

In this section, parameter estimates are derived using the maximum likelihood method. The MBXII-IE distribution, BXII-IE, Lomax-IE and Log-logistic-IE distributions are fitted to real data sets: fracture toughness, taxes revenue's data and coal mining disaster data for the comparison purposes. The log-likelihood function for the vector of parameters $\Phi = (\alpha, \beta, \gamma, \lambda)$ of the MBXII-IE distribution is

$$\ln L(\Phi) = n\ln(\alpha) + n\ln(\beta) + n\ln(\lambda) - 2\sum_{i=1}^{n}\ln x_i + \lambda\sum_{i=1}^{n}(x_i)^{-1} - (\beta+1)\sum_{i=1}^{n}\ln\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right] - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\ln\left\{1 + \gamma\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right]^{-\beta}\right\} \quad (23)$$

In order to compute the estimates of the parameters α, β, γ, λ of the MBXII-IE distribution, the following nonlinear equations must be solved simultaneously:

$$\frac{\delta}{\delta\alpha}\ell(\Phi) = \frac{n}{\alpha} - \frac{1}{\gamma}\sum_{i=1}^{n}\ln\left\{1 + \gamma\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right]^{-\beta}\right\} = 0, \quad (24)$$

$$\frac{\delta}{\delta\beta}\ell(\Phi) = \frac{n}{\beta} - \sum_{i=1}^{n}\ln\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right] + (\alpha+\gamma)\sum_{i=1}^{n}\left[\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right]^{\beta} + \gamma\right]^{-1}\ln\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right], \quad (25)$$

$$\frac{\delta}{\delta\lambda}\ell(\Phi) = \frac{n}{\lambda} + \sum_{i=1}^{n}(x_i)^{-1} - (\beta+1)\sum_{i=1}^{n}\frac{1}{x_i}\left[1 - \exp\left(-\frac{\lambda}{x_i}\right)\right]^{-1} + (\alpha+\gamma)\beta\sum_{i=1}^{n}\frac{1}{x_i}\left[1 - \exp\left(-\frac{\lambda}{x_i}\right)\right]^{-1}\left\{\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right]^{\beta} + \gamma\right\}^{-1}, \quad (26)$$

$$\frac{\delta}{\delta\gamma}\ell(\Phi) = \frac{\alpha}{\gamma^2}\sum_{i=1}^{n}\ln\left\{1 + \gamma\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right]^{-\beta}\right\} - \left(\frac{\alpha}{\gamma}+1\right)\sum_{i=1}^{n}\left\{\left[\exp\left(\frac{\lambda}{x_i}\right) - 1\right]^{\beta} + \gamma\right\}^{-1}. \quad (27)$$

Simulation Study

In this section, simulation study for the performance of the MLEs of the MBXII-IE parameters with respect to the sample size n is carried out. This performance is done based on the following simulation study:

Step 1: Generate 10000 samples of size n from the MBXII-IE distribution based on the inverse cdf method.

Step 2: Compute the MLEs for 10000 samples, say $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$ for i=1,2,...,10000 based on non-linearoptimization algorithm with constraint matching to range of parameters. (0.5, 0.5, 0.5, 0.9) and (0.8, 0.9, 1, 0.5) are taken as the true parameter values (α, β, γ, λ).

Step 3: Compute the means, biases and mean squared errors of MLEs.

For this purpose, we have selected different arbitrarily parameter values and n=50,100,200,300,500 sample sizes. All codes are written in R and the results are summarized in Table 2. The results clearly show that when the sample size increases, the MSE of estimated parameters decrease and biases drop to zero. As the shape parameter increases, MSE of the estimated parameters increases. This shows the consistency of the MLE estimators.

Sample	Statistics	α=0.5	β=0.5	γ=0.5	λ=0.9	α=0.8	β=0.9	γ=1.0	λ=0.5
n=50	Means	0.508	0.5275	0.4009	1.5208	0.8235	0.9994	1.1443	1.3154
	Bias	0.008	0.0275	-0.0991	0.6208	0.0235	0.0994	0.1443	0.8154
	MSE	0.0691	0.6688	14.5017	3.0359	1.0307	8.3995	318.6126	3.4638
N=100	Means	0.5047	0.5165	0.4233	1.1072	0.7974	0.9218	0.6404	0.8864
	Bias	0.0047	0.0165	-0.0767	0.2072	-0.0026	0.0218	-0.3596	0.3864
	MSE	0.0335	0.0542	0.1008	0.5377	0.2057	0.8449	2.629	1.2674
n=200	Means	0.5009	0.5092	0.4617	0.9831	0.7882	0.926	0.7403	0.6479
	Bias	9e-0.4	0.0092	-0.0383	0.0831	-0.0118	0.026	-0.2599	0.1479
	MSE	0.0177	0.0229	0.0401	0.1467	0.1524	0.1817	0.2849	0.2877
n=300	Means	0.5016	0.5071	0.4804	0.9509	0.7864	0.9253	0.8063	0.5879
	Bias	0.0016	0.0071	-0.0196	0.0509	-0.0136	0.0253	-0.1937	0.0879
	MSE	0.0117	0.0132	0.0266	0.0875	0.1211	0.1281	0.1785	0.1394

n=500	Means	0.5001	0.5029	0.4854	0.932	0.7921	0.9131	0.8687	0.5514
	Bias	1e-0.4	0.0029	-0.0146	0.032	-0.0079	0.0131	-0.1313	0.0514
	MSE	0.007	0.0071	0.0153	0.0481	0.0841	0.077	0.096	0.064

Table 2: Means, Bias and MSEs of the MBXII-IE distribution (0.5, 0.5, 0.5, 0.9) and (0.8, 0.9, 1, 0.5).

Applications

In this section, the MBXII-IE distribution is compared with BXII-IE, Lomax-IE, LL-IE distributions. Different goodness of fit measures like Cramer-von Mises (W), Anderson Darling (A), Kolmogorov- Smirnov (K-S) statistics with p-values, Akaike Information Criterion (AIC), consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC) and likelihood ratio statistics are computed using R-package for real data sets: fracture toughness, taxes revenue's data and coal mining disasters data. The better fit corresponds to smaller W, A, K-S, AIC, CAIC, BIC, HQIC and - l value. The Maximum Likelihood Estimates (MLEs) of the unknown parameters and values of goodness of fit measures are computed for MBXII-IE distribution and its sub-models.

Application I: (Fracture Toughness)

The values fracture toughness MPa m1/2 data from the material Alumina (23 Al O) are: 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35,5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5,4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

The MLEs, their standard errors (in parentheses) and goodness-of-fit statistics like W, A, K-S (p-values) are given in Table 3. Table 4 displays goodness-of-fit values.

Model	α	β	γ	λ	W	A	K-S p-value
MBXII-IE	0.0181133061 (0.022788744)	4.1601365928 (0.413218923)	0.0000000001 (0.002022918)	1.5161119726 (0.352629351)	0.1182952	0.7247703	0.0796 (0.4387)
BXII-IE	10.836337 (8.8614837)	2.939956 (0.5268154)	1	5.420295 (1.3394846)	0.2103398	1.295879	0.1186 (0.07024)
L-IE	88.15512 (30.158622)	1	1	20.65397 (1.627536)	0.3337133	2.026292	0.1147 (-0.08714)
LL-IE	1	(4.867383) 0.38248098	1	2.975707 (0.06835492)	0.4064326	2.522655	0.332 (8.12e-12)

Table 3: MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for fracture toughness.

Model	AIC	CAIC	BIC	HQIC	-l
MBXII-IE	346.419	346.7698	357.5355	350.933	169.2095
BXII-IE	351.0785	351.2872	359.4159	354.4641	172.5393
L-IE	356.4294	356.5328	361.9876	358.6864	176.2147
LL-IE	370.1331	370.2365	375.6913	372.3901	183.0665

Table 4: Goodness-of-fit statistics for fracture toughness.

The MBXII-IE distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for the MBXII-IE distribution.

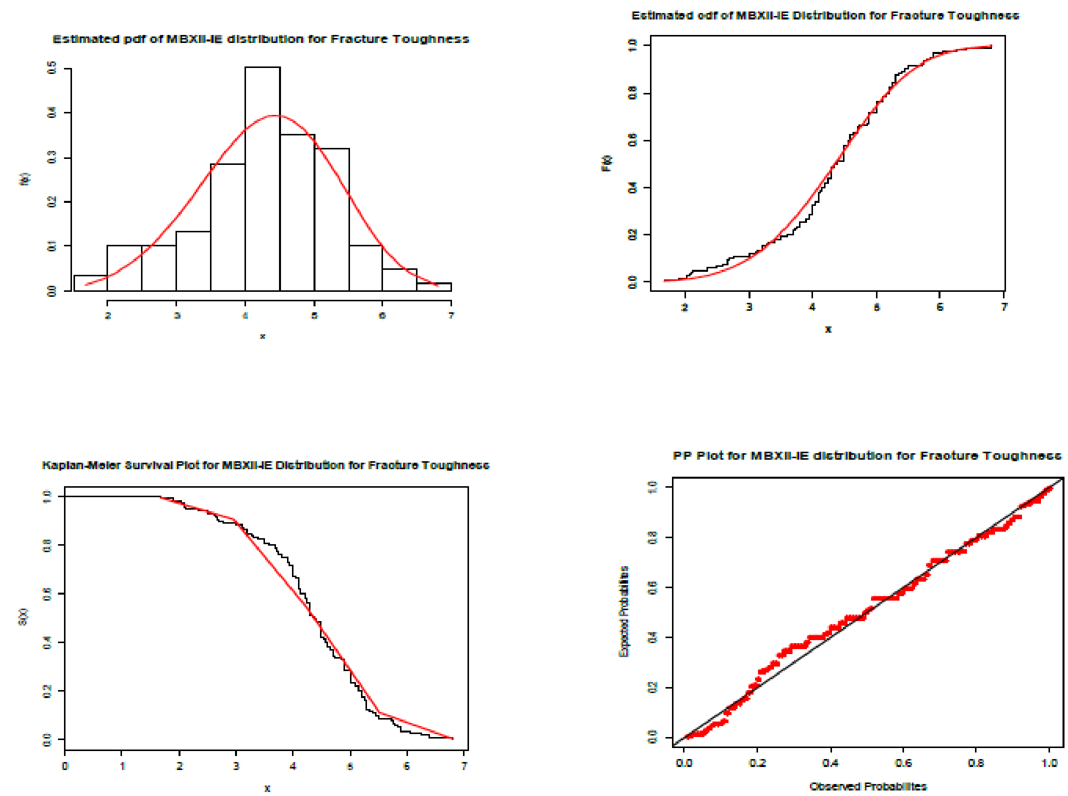

Figure 3: Fitted pdf, cdf, survival and pp plots of the MBXII-IE distribution for fracture toughness.

The new MBXII-IE model is much better than other three important competitive models with smallest value of AIC, CAIC, BIC, HQIC, K-S, W and A as well as largest p-value and –log likelihood value in modelling the first data set.

Application II (Taxes Revenue's Data): The values of tax revenue data are 3.7, 3.11, 4.42, 3.28, 3.75, 2.96,3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53,2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67,4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93,3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22,2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

The MLEs, their standard errors (in parentheses) and goodness-of-fit statistics like W, A, K-S (p-values) are given in Table 5. Table 6 display goodness-of-fit value.

Model	α	β	γ	λ	W	A	K-S p-value
MBXII-IE	0.01574783 (0.017879592)	2.66428346 (0.356379272)	0.00150912 (0.002280467)	0.54190197 (0.168851792)	**0.08121186**	**0.4526733**	**0.1009 (0.2603)**
BXII-IE	3.405735 (1.5676440)	1.865183 (0.3395644)	1	2.831844 (0.6706193)	0.2254017	1.182635	0.2072 (0.0003741)
L-IE	9.196698 (2.1114083)	1	1	6.214104 (0.5991982)	0.3149302	1.693847	0.1703 (0.006058)
LL-IE	1	2.770423 (0.23340747)	1	1.676333 (0.07349948)	0.3281241	1.739495	0.3634 (6.722e-12)

Table 5: MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for Tax Revenue.

Model	AIC	CAIC	BIC	HQIC	-l
MBXII-IE	289.7468	290.1678	300.1675	293.9642	140.8734
BXII-IE	300.7174	300.9674	308.5329	303.8805	147.3587
L-IE	304.8322	304.956	310.0426	306.941	150.4161
LL-IE	307.8828	308.0065	313.0932	309.9915	151.9414

Table 6: Goodness-of-fit statistics for Tax Revenue.

The MBXII-IE distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for the MBXII-IE distribution.

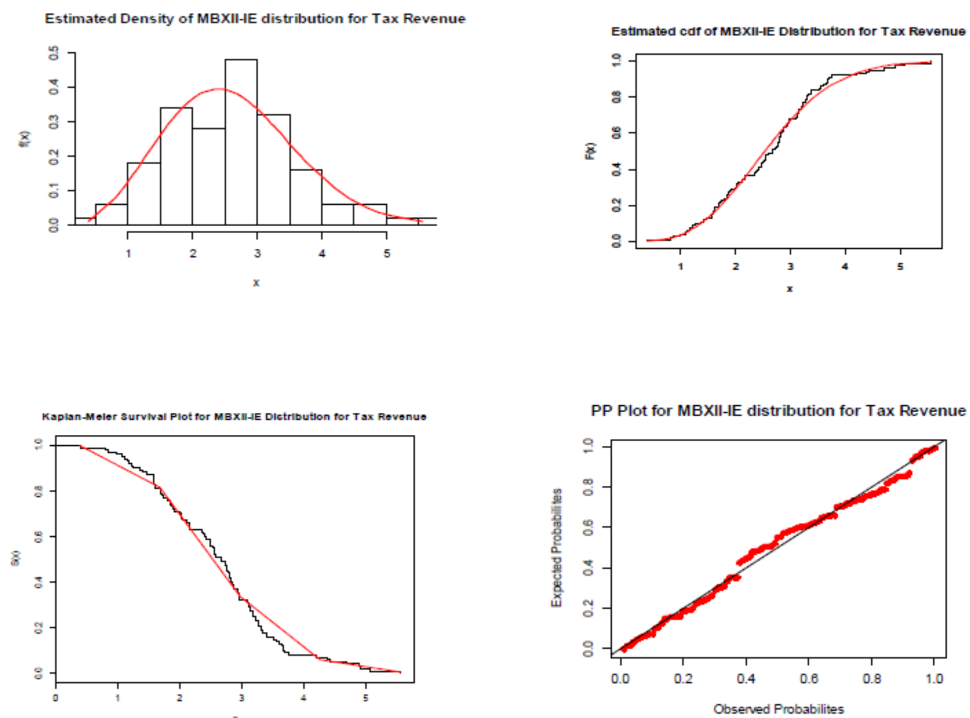

Figure 4: Fitted pdf, cdf, survival and pp plots of the MBXII-IE distribution for Tax Revenue.

The new MBXII-IE model is much better than other three important competitive models with smallest value of AIC, CAIC, BIC, HQIC, K-S, W and A as well as largest p-value and -log likelihood value in modeling the second data set.

Application III: (Coal mining data set) published the data related to intervals in days between 109 successive coal mining disasters in Great Britain during the period 1875-1951. The values of data are 1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36,37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66,72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120,123, 124, 129, 131, 137, 145, 151, 156, 171, 176,182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312,315, 326,326, 329, 330, 336, 338, 345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871,1312, 1357, 1613,1630.

The MLEs, their standard errors (in parentheses) and goodness-of-fit statistics like W, A, K-S (p-values) are given in Table7. Table 8 display goodness-of-fit values.

Model	α	β	γ	λ	W	A	K-S p-value
MBXII-IE	0.014003462 (0.010293280)	0.974040069 (0.114284919)	0.002542779 (0.001955081)	2.319100102 (1.400766759)	0.06743529	0.4651009	0.1142 (0.1167)
BXII-IE	0.5257374 (0.1056558)	0.9798724 (0.1313540)	20.8102228 (4.3210242)	1	0.804424	4.659193	0.5874 (<2.2e-16)
L-IE	0.5129513 (0.0608668)	1	1	20.4062733 (3.3094989)	0.8043515	4.653696	0.592 (<2.2e-16)
LL-IE	1	0.6939298 (0.05621336)	1	36.3055045 (4.43815735)	0.9376481	5.457812	0.521 (<2.2e-16)

Table 7: MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for coal mining disasters.

Model	AIC	CAIC	BIC	HQIC	-l
MBXII-IE	1409.089	1409.474	1419.854	1413.455	700.5445
BXII-IE	1493.264	1493.493	1501.338	1496.539	743.6322
L-IE	1491.287	1491.4	1496.67	1493.47	743.6435
LL-IE	1503.023	1503.136	1508.406 1	505.206	749.5115

Table 8: Goodness-of-fit statistics for coal mining disasters.

The MBXII-IE distribution is best fitted model than the other sub-models because the values of all criteria of goodness of fit are significantly smaller for the MBXII-IE distribution in Figure 5.

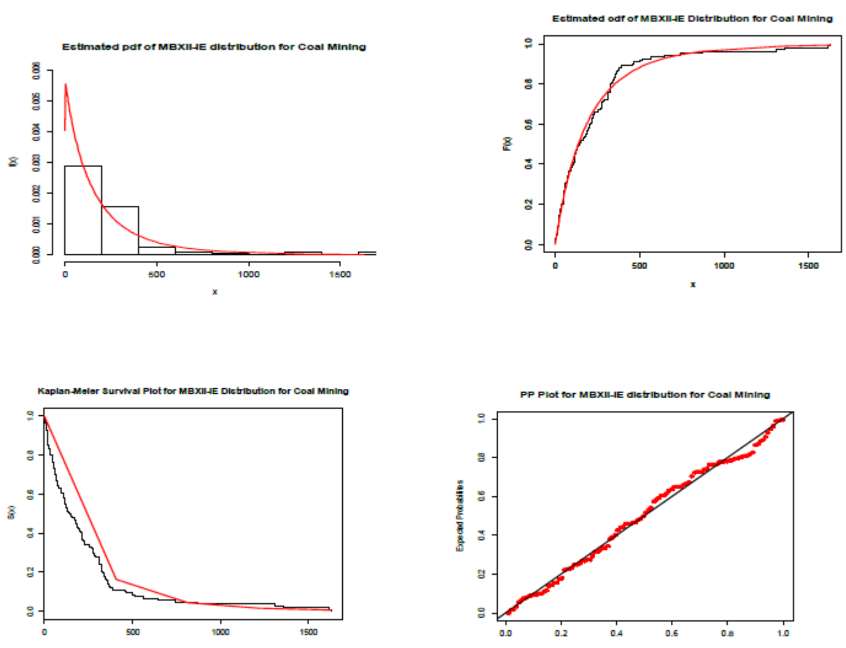

Figure 5: Fitted pdf, cdf, survival and pp plots of the MBXII-IE distribution for coal mining disasters data.

The new MBXII-IE model is much better than other three important competitive models with smallest value of AIC, CAIC, BIC, HQIC, K-S, W and A as well as largest p-value and -log likelihood value in modeling the third data set.

Concluding Remarks

We have developed the MBXII-IE distribution along with its properties such as: descriptive measures based on the quantiles, moments, stress-strength reliability and multi component stress-strength reliability model. The MBXII-IE distribution is characterized via different techniques. Maximum Likelihood estimates are computed. The simulation study is performed using the MBXII-IE distribution to illustrate the performance of the MLEs. Goodness of fit shows that MBXII-IE distribution is a better fit. Applications of the MBXII-IE model to fracture toughness, taxes revenue's data and coal mining disasters data are presented to demonstrate its significance and flexibility. We have shown that the MBXII-IE distribution is empirically better for fracture toughness, taxes revenue's data and coalmining disasters data.

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APPENDIX A

Theorem1. Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a given probability space and let $\mathrm{H} = [a_1, a_2]$ be an interval with $a_1 < a_2$ ($a_1 = -\infty, a_2 = \infty$ are allowed). Let $X : \Omega \rightarrow H$ be a continuous random variable with distribution function F and Let $g(x)$ be real function defined on $\mathrm{H} = [a_1, a_2]$ such that $E[g(X) | \mathrm{X} \geq \mathrm{x}] = h(x)$ $x \in H$ is defined with some real function $h(x)$. Assume that $g(x) \in C([a_1, a_2])$, $h(x) \in C^2([a_1, a_2])$ and F is continuously differentiable and strictly monotone function on the set $[a_1, a_2]$: Assume that the equation $g(x) = h(x)$ has no real solution in the interior of $\mathrm{H} = [a_1, a_2]$. Then F is obtained from the functions $g(x)$ *and* $h(x)$ as $F(x) = \int_a^x k \left| \frac{h'(t)}{h(t) - g(t)} \right| \exp(-s(t)) dt$, where $s(t)$ is the solution of differential equation $s'(t) = \frac{h'(t)}{h(t) - g(t)}$ and k is a constant, chosen to make $\int_H dF = 1$.