

Review Article

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Brief Historical Tour of Glacier Ice on Earth and its Role in Climate Dynamics¹

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Abstract

The science of the physics of ice crystals started almost 500 years ago with Cardano, followed by Kepler, Hooke, Dalton, the two Braggs (father and son) and Pauling. Here, we are concerned with naturally formed ice of glaciers, ice sheets and ice shelves. We do not discuss lake, river, sea and atmospheric ices, even though they equally play a role in today's General Circulation Models (GCMs) for the Earth.

The study of glacier ice started with the observation of the behavior of the dynamics of Alpine glaciers, specifically that they are not rigid, but moving bodies that deform. A ladder, left in 1788 at the icefall of the Col de Géant by de Saussure was 44 years later found at the three-glacier merge of the Mer de Glace, corresponding to a mean velocity of 375 feet/year. Hugi and Agassiz measured the motion of rocks on the middle moraine of the Unteraar-gletscher and found similar values. However, at early times their location of origin was an enigma; so they were named erratic. Forbes and Tyndall found by position measurements with theodolites that velocities of glacier surface objects are (i) considerably larger in summer than in winter; (ii) larger on the glacier surface than at depth and (iii) larger in the middle of the glacier-width than close to the boundaries, as reported by Helmholtz (1865), [77]. On this basis Rendu and Forbes were the first to identify similarities of glacier flows with streams of very viscous fluids. It lasted until the 1950s when physicists had postulated the constitutive equation for isotropic ice as a non-Newtonian power law fluid (Nye 1952), [134], that was experimentally verified by Glen (1952, 1953), [55,56], and Steinemann (1954, 1956, 1958), [157-159]. Orowan (1949), [136], likely prompted the British contributions. This law is now known as Glen's flow law. Various second grade fluid alternatives (e.g. Man and others (1985, 1987, 1992 2010), [116-118], provide extensions to capture primary and secondary creep. Comparing results of creep tests from distinct experimentalists disclose unexplained disparities of stress relations.

All large-scale mass Initial Boundary Value Problems (IBVPs) are viscous, heat conducting incompressible Stokes problems, using the Glen-Steinemann flow law, which may involve cold ice regions with the ice temperature below melting, separated (close to the basal boundary) by a Class-I mixture of temperate ice, i.e. ice with water inclusions. The dynamics of the cold-temperate transition surface between the two regions is governed by a (simplified) Clausius-Clapeyron equation (Fowler & Larson (1978, 1980), [49-50] Hutter (1980, 1983), [84-86], Hutter, et al. (1981, 1988), [82, 88], [Blatter & Hutter (1991), [15]]).

As for boundary conditions at the free surface, snow accumulation and radiative heat must be parameterized as functions of position and time or these quantities must be taken over from concurrent large-scale flow models of the atmosphere. One must proceed in a similar way with the ice ocean interface on floating portions of ice shelves. Here, general ocean-circulation models are operative. Moreover, at the ice-shelf front, mass loss of ice by calving must be parameterized, which is significant for marine glaciers, ice shelves, and their dynamic response in such scenarios.

Motivation

To begin, let us briefly present those articles that treat some aspects of the historical development of the physics of glacier flow. We collect articles in which the focus is devoted to the history of glacier and ice sheet flows approximately during the past 3 centuries but also including the past 30 years. Our focus is the Earth bound slow movement of ice masses and how this creeping

¹**Acknowledgment:** Dr. M. Huss commented on an earlier draft and Dr. H. Zellokari performed a literature search on computational ice sheet models of the 21st century in which SIA, SSA and Stokes models were employed. Michael Imhof allowed us to see the forthcoming paper [96], which helped us to streamline our manuscript. Prof. D. MacAyeal. Prof. F. Navarro, M. M. Magnússon and Prof Kirchner, N. provided support. We appreciate all these supports and express our thanks.

flow was, in earlier centuries, intellectually developed and principally rationalized, eventually as a nonlinear viscous heat-conducting fluid. Of similar significance as the early material science properties of ice was the recognition of the concept of ice age(s). For the 19th century scientists, the enigmatic appearance of erratic boulders led to the recognition of the concept of ice age(s). This concept, first spelled out around 1830 by the German poet Goethe, see Cameron, D. (1965), [24], was only formally rationalized as the planetological cause of the ice ages that is coupled with Earth internal processes affecting the time evolution of the Earth's climate.

The most significant memoir, in which the physics of glaciers is reviewed, is Garry Clarke's (1987) 'A short history of scientific investigations on glaciers', [27]. The author gives a tour of 'all' glacier related processes, starting in the first quarter of the 19th century, but primarily focusing on the years ~1940-1987 and describing mostly research performed in the past 50 years related to the physics of glaciers. This work is much geared to accumulate the varied knowledge separated by subjects; it is excellently summarized. Its 19th century coverage is similar to ours, but with a broader coverage of subtopics and has perhaps less of a natural-philosophical emphasis than ours.

The paper 'Decent of glaciers: some early speculations on glacier flow and ice physics by John Walker and E.D. Waddington (1988), [171], concentrates on the early work of the motion of glaciers as observed in the 19th century. They cover the literature of the period from ~1820 to 1870, notably listing the memoirs of Agassiz (1837, 1840, 1842), [3-5] de Charpentier (1841), [26] Forbes (1841, 1842, 1843), [45-47], covering essentially what was stated by Clarke (1987), [27].

The memoir 'Life, death and afterlife of the extrusion flow theory' by E.D. Waddington (2010), [170], is devoted to a specialized subject, namely extrusion, which was a fallacious model of ice flow. In our manuscript, we do not deal with such phenomena, i.e. processes that failed to be recognized as realistic.

Two additional memoirs in the *Journal of Glaciology* are devoted to the hydraulic component of Alpine glacier flow. Gwenn Flowers (2010), [43], devotes her paper to such related works by Almut Iken (1933-2018), [93-95]. Similarly, Christina Hulbe, et al. (2010), [81], devoted their paper to Women in glaciology. Both these articles are at best tangibly related to our present concern of glacial flows.

Thus, it transpires that our focus is, first, the birth of the description of glacier ice flows as the motion of a heat-conducting fluid and, second, the presentation of the necessity of extending the flow law of ice from its nascent postulation beyond the Glen-Steinemann flow law as a general nonlinear viscous fluid. We stress circumstances where we believe that present research activities do not strictly enough draw the inferences that should be drawn on the basis of the findings.

To understand how the large ice masses and glaciers on Earth contribute to the climate scenarios, one must first securely know to what extent the climate driving of the solar system interacted, and will interact, with the Earth-interior processes, and how these affect the various components: the atmosphere, the solid Earth, the oceans and hydrosphere, as snow, ice, water and vapor, not to forget the biosphere, which also has a significant anthropogenic component. Unquestionably, there is also an anthropogenic component, as demonstrated by industrial activities since approximately the year 1780 and expressed by the equivalent CO₂ concentration in the atmosphere, about 280 ppm (parts per million) in the year 1780 and 415 ppm on May 19. 2019 as measured by the Mauna Loa Observatory in Hawaii.

This brief account of the most significant components of the Earth's climate system only touches those parts of the entire picture, as they chiefly developed in the last 90 years or so, beginning with Milankowitch's thesis (1930), [124]. It consists of three distinct but concurrent processes: eccentricity, obliquity, and precession. Eccentricity, the elliptical cycle of variation in shape of the Earth's orbit around the Sun is about a 100.000-year cycle. Obliquity is the cycle of axial tilt of the Earth's rotation axis toward or away from the Sun and varies from about 22 to 24.5 degrees, usually taken as 23.5 degrees with about a 26.000-year cycle. As these parameters change, so does the amount of sunlight that hits different latitudes on the Earth; for a NASA illustration of these, see Gustovich (2018), [74].

As a scientific focus, climate dynamics on Earth and its anthropogenic coupling came last. In the period from the mid-17th to the mid-19th century, glaciers were first thought to be rigid objects. That they somehow move was mainly recognized in the 18th century, in parts by conjecturing that the volume expansion of water in the freezing process would push the ice downward in their valleys. Later in the 18th century this belief was replaced by the hypothesis that glaciers would slide over their beds. Moreover, edgy rocks and dirt on the glacier surface, which were seen to move, brought the impetus for the assumption that glaciers move and deform. Careful geodetic measurement with theodolites in the 18th and 19th century, finally, disclosed the creeping character of this motion as deformable like a dough. A further enigma was still the inexplicable presence of erratic boulders within an environment of unmatchable petrography. The solution of this mystery led finally to the hypothesis of earlier ice ages, a fundamental topic of climate dynamics, postulated in the 18th and 19th century and rationally explained in the 20th century.

To be able to describe the deforming motion of glaciers calls for material science of the ice. The determination of the material properties of polycrystalline ice as a nonlinear viscous, heat-conducting body - at early times often denoted as plastic - is vital for the quantification of its melting processes. This concerns the substance H₂O and its appearances as ice, water and vapor. These are

described by the thermodynamic principles (first and second law). It so happens that the Earth's climate conditions through the past few million years have driven the phase changes of H_2O so sensibly that these processes reign not only the various forms of life on Earth, but also its ice coverage, the evolution of the sea level due to natural external variations as well as anthropogenic Earth internal climate changes. These latter will be the cause of roughly 80 cm sea level rise until the end of the 21st century, according to the IPCC (Intergovernmental Panel on Climate Change) Report.

This brief overview of how the phenomena 'glaciers and ice sheets' triggered the curiosity of mankind to eventually become a respected part of the natural sciences of even high significance for the humans' living on this planet. This picture must be complemented by the material sciences. These ought to describe the H_2O phases as objects of mathematical physics. Only with the addition of the description of the material behavior of ice, the ice processes on Earth can rationally be described; only with these, quantitative predictions can be made. This phase begins with the recognition that isotropic polycrystalline ice under slow creep can be modeled as a power law heat-conducting fluid. It goes back to experiments conducted in the 1950s by John Glen (1952, 1953), [55, 56], and S. Steinemann (1954, 1956, 1958), [157-159]. This law reads for simple shear

$$\dot{\gamma} = f(T, \tau^2)\tau,$$

where $\dot{\gamma}$ is the rate of strain (rate of the shear angle), T is the temperature and τ is the shear stress. The scalar function f is generally written as the product

$$f(T, \tau^2) = A(T)(\tau^2)^{(n-1)/2}. \quad (1)$$

$A(T)$ is called rate factor, and n (generally chosen as $n = 3$) determines the exponent of the power law. In the glaciological community, the law (1) is called Glen's flow law (we call it the Glen-Steinemann flow law, because the law was approximately simultaneously and independently proposed in the 50s of the last century by both researchers in the UK and Switzerland, respectively²). However, in material science, depending on the field of science, it is attributed to different distinguished scientists, in rheology to Graham (1850), Guoy (1910). Ostwald-de Waele³ brought it forward in (1929) in the context of colloidal fluids and Reiner already before (1929) in rheological problems. In metallurgy, it is called Norton's flow law (1929), and in plasticity theory Orowan had used it. In Glaciology John Nye (1953), [134], had demonstrated to the applied glaciologists, how it could be 'derived' from simple material postulates of viscous fluid behavior. It is likely that Orowan, who had also worked on constitutive behavior of plastic materials at that time, had directed Glen and Nye into such a nonlinear law; all three were then working at the Cavendish Laboratory at Cambridge University, UK, where Glen and Nye were junior collaborators.

So, the Glen-Steinemann flow law has precursors in not too distant related fields. We scrutinize its performance in comparison to laboratory and in-situ experiments and will also report on a number of attempts to the parameterization of the proposed laws. The likely best form of this viscous thermo-mechanical material description will then at last be incorporated in full-scale computations of glaciers and large ice masses in regional and completely Earth-embracing physical-mathematical-numerical models. This is in order to try to understand the climate relevant processes on our Globe for computational reproduction of the past climate variation due to the thermo-mechanical input data from extra-Earth processes, as well as to predict of future reaction.

Early Deformation and Sliding Models of Alpine Glaciers

The proposals for the enigmatic large ice masses by Johann Jakob Scheuchzer⁴ and others in the 18th century can only be understood, if we accept that prior to the mid-19th century no quantification of constitutive relations was available to the material

²The authors were repeatedly criticized by the attribution of Glen's flow law to Glen-Steinemann and the arguments of this critique seem to be the facts that Glen published the power law prior to Steinemann, according to these 'experts'. However, the documented facts do not necessarily transmit the historical facts. True is that Steinemann's Ph. D. dissertation was completed around 1953, i. e. at the same time as Glen's, and then submitted, but sitting on the professor's shelf, collecting dust for about 5 years, when it was published in 1958 (in the German language): *Experimentelle Untersuchungen zur Plastizität von Eis*, Beiträge zur Geologie der Schweiz, Nr. 10, 1958) in a geological periodical of the Swiss Natural Sciences. Steinemann managed to publish some excerpts between 1952 and 1958 (in the English language); these are the papers that are usually referred to in the literature. KH has not seen the 1958-paper being generally referenced internationally. Steinemann left glaciology and went to Lausanne as a professor of physics. He might well have been fed-up with the German-Swiss environment in the 1950s. However, Glen took-up Steinemann's experimental findings in shear-compression experiments and generated the evidences of the failure of the power law in his 1958-paper. In this context it is worth mentioning that the first paper on the power law of ice is written by Orowan [136], and Glen & Perutz, [54], cite Steinemann's power law already in 1954. Moreover, also in a series of other papers the (power) flow law is attributed to both Glen and Steinemann, see e.g. Budd et al. (2013) [17]. We might also add that there is a "Steinemann Island" close to the Antarctic Island, named by UK Antarctic Place-names Committee (UK-APC) in 1960 for Samuel Steinemann, Swiss Physicist, who made laboratory investigations on the flow of single and polycrystalline ice.

³As an example of the many fundamental articles on this subject we restrict here mentioning that of W. Oswald in 1929 [137].

⁴ Johann Jakob Scheuchzer (02.08.1672 – 23.06.1733) physician and natural scientist from (and in) Zürich, Switzerland. He studied medicine in Altdorf, near Nürnberg and in Utrecht, where he received his medical doctorate. In the same year, he took his first journey to the Alps. In the year 1695, when one of the four physicians of the town Zurich had passed away, the Zurich government appointed him as his successor.

scientists interested in the physics of large ice masses. Nevertheless, glaciers as natural phenomena at high altitudes of mountainous regions were attractive, if simply for their mystic appearance. Prior to the 17th century, not even the deformability of glacier ice was recognized, certainly not admitted. Glaciers were postulated to be rigid objects, see e.g., Moraltus, Johann (1669), [105]. Only valley inhabitants close to them accepted some deformability, because e.g. they observed their snouts to advance or retreat, but could hardly rationalize a cause, and if they had been able to, they were too isolated in their valleys to get in contact with musing intellectual hikers. These modest people often knew it better.

So, with this background in the year 1705, Johann Jakob Scheuchzer visited Swiss Glaciers and proposed a theory on their motion. He knew from physics that water is expanding in the freezing process to ice, and that the 'Force' of expansion is so large that cartridge rounds which are filled with water that freezes, are blasted into pieces. Scheuchzer assumed that the water in glacier fissures and crevasses that freezes would extend with such excessive power that its force will unquestionably push the glacier downward. This concept, later often called 'Dilatation Theory', was also adopted by Jean de Charpentier⁵, his brother Toussaint de Charpentier⁶ and Louis Agassiz⁷ and others. Basic thought of this concept was the belief that glaciers are permanent storehouses of coldness, capable to freeze all water that percolates through them. Interesting and strange in this model is that the volume expansion in the freezing process of ice is the only cause of the downward motion of the ice of glaciers. As a question at hindsight one might ask whether these scientists had ever looked at the water outlets of Alpine glacier termini with their sub-glacier rivers and generally substantial water discharge that is often rather large, certainly not zero during summer, as they required by their early deformation postulate.



Figure 1: 'Gletschertor' (water outlet) of the Anengletscher, Switzerland (Photo Johannes Löw).

About in the year 1760, Altmann⁸ & Gruner⁹ brought forward their opinion that glaciers would move by means of sliding along their beds. Both published their contributions independently, Altmann in 1751 and Gruner in 1760. Almost 40 years later, Horace

Scheuchzer's merits are wide embracing: he replaced the trigonometric determination of the heights of positions on the Earth by much more accurate barometric instruments; with his 'Herbarium divularium' he became the founder of paleo-botany. In the year 1713 he drew with his 'Nova Helvetiae Tabula Geographica' the best and most accurate map of Switzerland at that time. In several journeys through Switzerland he also visited in 1705 the 'Rhône Glacier', and wrote a report, which also contains the other Swiss glaciers as they were known at that time.

⁵ Jean de Charpentier (07.12.1786, Freiberg (Saxony)-12.09.1855, Bex, Switzerland) was a Swiss geologist; he was first mining engineer in Silesia, worked then in the Pyrenees where he wrote several geological memoirs. In the year 1813 the government of the Canton Vaud, Switzerland, offered him the directorship of the salt mine in Bex where he stayed until his death.

⁶ The older brother of Jean, Toussaint de Charpentier (1779-1847), equally a geologist, mining engineer and entomologist also adopted this phase change argument as cause for the forward glacier motion.

⁷ Louis Agassiz (28.05.1807, Neuchâtel-14.12.1873, Cambridge, Mass., USA), a descendant of a clergyman, was a leading paleontologist and natural scientist of the 19th century focusing on fish-fossils and glaciology. He received his higher education from the universities of Zurich, Heidelberg and Munich in medicine and natural sciences and received his doctorate in philosophy from the University of Erlangen and his medical doctorate from the University of Munich. He was professor at the University of Neuchâtel and (later at Harvard University, Boston, Mass. USA, where he stayed until his death). Since 1836, Agassiz was a leading glaciologist. He, with his research group, performed in 1840, see [4], measurements on the Aare-Gletscher, where they built a hut ('Hôtel des Neuchâtelois'), focusing on the properties of glacier ice, its temperature and the water circulation as well as the motion and mobility of the ice. He was one of the first to postulate the existence of the Ice Age, but interpreted it as a sudden event that preceded the rise of the Alps. The German poet Johann Wolfgang von Goethe anticipated the postulation of the Ice Age in 1831 (see Cameron, D., (1965), [24])

⁸ Johann Georg Altmann (21.04.1695, Zofingen – 18. 03.1758, Ins) studied at the Theological School in Berne, Switzerland, before he worked during 1725/26 and again 1734/35 as protestant clergyman in Wählern. From 1734 to 1757 he taught in Berne as Professor of eloquence and ancient Greek and ethics. His 'Versuch einer historischen und physischen Beschreibung der helvetischen Eisbergen' ('Attempt of a historical and physical description of the Helvetic icebergs'), edited by Heidegger and Compagnie, Zurich 1751, brings forward his sliding postulate of glacial movement. (Available as a pdf at ETHZ library)

⁹ Gottlieb Sigmund Gruner (20.07. 1717, Trachselwald, Berne--10.04. 1778, Utzensdorf) was born into a Bernese patrician family and grew up in the town of Burgdorf in the Canton of Berne. After studying law he was employed as archivist for the Landgrave of Anhalt-Schaumburg and

Bénédict de Saussure revived this concept, which became the ‘de Saussure Theory’, even though, according to Tyndall (1878, p 185), he was never chiefly involved in it.

Cautious Birth of the ‘Plasticity Theory’

The 18th century also brought the first ideas about the deformability of ice sitting on solid ground. The concept of sliding should simply have brought the science mountaineers to the postulation of the deformability of glacier ice. Indeed, glaciers generally mostly move on non-planar beds; this fact, together with the existing forward motion requires deformability to maintain the sliding hypothesis. Yet, none of the above mentioned scientists attributes the terms ‘viscosity’ and ‘kneadability’ to characterize the deformability of the ice, even though, according to Tyndall (1878, p. 185), [164], the appearances of many glaciers suggest these terminologies, if it were not so contradictory to any daily experience with ice. Needless to say that the concept of viscosity was known since Newton had introduced it in the Principia, (first edition 1687, third edition 1726), [132].

In spite of this, these kinds of plastic concepts found their defenders. In the year 1773, André César Bordier from Geneva published a small booklet ‘Voyage pittoresque aux Glaciers de Savoie’, Genève, 1773 (‘Picturesque journey into the Glaciers of Savoy’). He advocates for an over-all view of the ice mass which moves as a whole from high to lower altitudes in a manner as seen with other fluids. ‘Do not let us view the ice as an immobile and stiff material, similar to mollified wax, which, to a certain degree, is flexible and extensible’. This is likely the first occasion where the kneadability of glacier ice is mentioned.

However, according to Tyndall, André César Bordier’s¹⁰ concepts were unheard in the natural scientific community in the 1770s. They were reborn more than 60 years later by a successor of better scientific training. This person was a catholic priest and later bishop of Annecy, Louis Rendu¹¹. In the year 1840 he submitted ‘Théorie des Glaciers de la Savoie’, [146], to the Royal Academy of Savoy. Tyndall, was fascinated by Rendu’s writings and states that it is not known, whether he ever saw Bordier’s works, probably not, because he never refers to it. Tyndall quotes a few of Rendu’s statements to illustrate the preciseness of his expressions:

- Between Mer de Glace and a river, there exists such a perfect similarity that it is impossible to find a circumstance in a glacier, which would not equally occur in a river.
- In flows of water, the movement is not uniform, neither with regard to the width nor to the depth.
- The friction at the bed and at the sides, paired with the effect of obstructions, makes the motion to differ from position to position; only toward the middle of the free surface, one reaches the full motion.
- There exists a large set of facts, which seem to force us to believe that glacier ice possesses some sort of extensibility, which allows it to adjust to the local circumstances, to thin, to swell and to contract as if it were a soft dough.

To corroborate these inferences, Rendu requested careful measurements of the glacier motion; he did not perform these himself, but asked his mountain guide to observe the motion of a particular boulder at the glacier side, which, during 5 years moved 40 feet/year. Other boulder position measurements corroborated the motion. This is a first indication to search for motion patterns of glaciers, which we shall subject to a detailed analysis later on.

To summarize

First descriptions of the motion and deformation of glaciers are due to Bordier in 1773, but the flexibility and extensibility with a characterization of viscous or plastic behavior was recognized by Rendu as late as 1840, [143], and afterwards, but could not yet be phrased in terms of a constitutive relation such as the Glen-Steinemann law.

Erratic Boulders

Since the 19th century, the edged boulders, apparently incoherently distributed in mountainous territory are named glacial erratic, which explicitly suggests for us that they were once moved by glaciers and deposited at their present positions when the carrier ice retreated. These rocks suggested to the mountaineers misleading or inexplicable or false behaving, briefly err-behaving. The causal connection had first to be recognized, and it influenced at least subconsciously the early understanding of the existence of

travelled as such through Prussia and Silesia. In 1779 he permanently returned to Switzerland, where he was employed as an attorney in the Canton Berne. Gruner had broad interests that also included mineralogy and geology. He wrote a three-volume treatise ‘Die Eisberge des Schweizerlandes’ (‘The ice mountains of Switzerland’) that appeared in 1760/62. In this treatise, he presented his theory of sliding of glaciers. (This is an excerpt from Tobias Krüger, 2009), [105].

¹⁰ André César Bordier (24.12.1746, Geneva – 18.03.1802, Céigny) studied protestant theology prior to 1770 and was a clergyman from 1770 - 1774. In 1775 he became a member of the ‘Council of the 200’ in Geneva, in 1777 he became the first judge at the court martial, 1790 councilor and in 1791 syndic trustee of Geneva. In 1794 he was (in absence) sentenced to death by the revolutionary court of justice of Geneva.

¹¹ Louis Rendu (09.12.1789, Meyrin – 08.08.1859 Annecy) was a French roman-catholic priest and later bishop of Annecy and a scientist. He was the author of ‘Théorie des glaciers de la Savoie’, an important book on the mechanisms of glacial motion. The Rendu Glacier, Alaska, U.S. and Mount Rendu, Antarctica are named for him, see also Hollier, J. & Hollier, A. (2016), [79].

the Ice Ages. According to Krüger (2009), [105], the oldest written statement on erratic boulders dates from 1301 and is reported by 'Henricus dictus von dem Steine'¹². The first assured written statement was found in Johannes Guler's 'Raetia'¹³. In his script Krüger (2009), [105] tells that Guler von Wyneck mentions a colossal rock in Veltlin, in the Italian Alps, for which he could not see, where it may have broken off. With the growing development of modern geology in the mid-18th century the general interest in the erratic boulders grew. In 1727/28 Moritz-Anton Capeller¹⁴, physician of the town Lucerne, recognized the alpine origin of the erratic boulders, deposited in the mountainous foreland. However, he only published his recognition in 1767 in his 'Pilatii montis historia', shortly before his death.

Initially, the contemporary scientists as e.g. the notary Abraham Schellhammer¹⁵ (1675-1735) and Albrecht von Haller¹⁶ both from Berne assigned the ultimate cause of the positional spreading of the erratic boulders to the biblical deluge or (later) to a number of exceptional floods. Tobias Krüger (2009), (2013), [106, 107], reports a number of exceptional and extreme interpretations, even by scientists of the status of Horace Bénédict de Saussure (1740-1799) and many others.

One interpretation by the Prussian geologist Christian Leopold von Buch¹⁷, why an erratic boulder found in the Jura Eastern slopes, but of likely origin in the upper Valais is as follows: He states that originally far in the past, today's Rhone Valley was blocked at Saint Maurice by a rocky Rigel (bedrock bar) between the Dents du Midi and the Dents des Morcles. Behind, a gigantic lake was formed up to the mountain peaks. At the breakdown of this natural Rigel, the water masses were set free with hardly imaginable power; they threw huge rocks as far as to the Jura (Krüger (2009), [106], notes that this information was only published after von Buch's death in 1867). We will refrain here from reporting further erroneous propositions of the displaced positions of erratic boulders. Interesting, however, is that in the mid-18th century, more than 100 years earlier, in 1742, the engineer and geographer Pierre Martel (1701-1767) from Geneva, reports of large displaced rocks found during his journey to the Valley of Chamonix. Its inhabitants had apparently explained to him that the 'Glacier du Dois' once carried these rock pieces down the valley. Krüger (2009), [106], concludes: For this reason, at the present state of knowledge [meaning 2009], the inhabitants of the Savoy Alps were the first, who established a causal relation between glaciers and position-alien rocks.

In the last decade of the 18th century, James Hutton¹⁸ devoted his working time to geology and authored the treatise 'Theory of the Earth', originally planned as four volumes. He adored Saussure's imagination in his theory of erratic boulders, but rejected Saussure's concept that erratic boulders reached their places of final position before the erosion had formed today's valleys. He writes 'there would then have been immense valleys of ice sliding down in all directions towards the lower country and carrying large blocks of granite to a great distance where they would be an object of admiration after ages, conjecturing from whence or

¹² State Archive Solothurn (Switzerland) :ISIL: CH-000043-5, document collection, document of 5. Oct, 1301, according to [105].

¹³ Guler von Wyneck, Johannes (31.10.1562, Davos – 03.02.1637, Chur, both in the SE of Switzerland) received his first education in the Latin school of Chur and studied then in Zurich, Basel and Geneva. Because he was in 1582 elected as local governmental secretary ('Landschreiber', in German), he could not complete his studies. In the year 1587 he became ruler ('Landeshauptmann', in German) of the Veltlin. In his second marriage, he was married to Elisabeth von Salis, a descendent of the very influential family in Graubünden, which brought to him the noble attribute 'von Wyneck'. He was politically very influential, and furthered the formation of an alliance of the Drei-Bünden (three confederations), but left to Zurich to escape local political disturbances. He then withdrew from political activities for more than a decade. His merits as a historian and mapmaker are exceptional, evidenced by his chronograph 'Raetia'.

¹⁴ Moritz-Anton Capeller (09.06.1685, Willisau–16.09.1769, Beromünster, both in Switzerland) grew up in Lucerne, where, after passing the Latin school, he completed his education in the Jesuit college. From 1700-1704 he studied mathematics and philosophy at the Collegium Helveticum in Milano; the medical studies were completed in the year 1706 in the Lothringian Academy at Pont à Mousson. During the Spanish war, he served as physician and engineer in Naples. He returned 1710 to Lucerne to follow his father as a private physician of the town. He served as member of the Lucerne Town Parliament and also had some assignments as an engineer (e.g. correction of the torrents) and acted as a teacher of mathematics and geometry at the local school of artillery. He earned scientific recognition and fame by his crystallographic-mineralogical works ('Prodromus crystallo-graphiae') that brought him the membership in the Royal Society of London. His principal work is on the history of the mountain Pilatus ('Pilatii montis historia', 1723-1728). His interests were wide, and so were his correspondences e.g., with Jakob Scheuchzer and Albert von Haller.

¹⁵ Abraham Schellhammer assigned the distribution of such orphane rocks to the biblical flood in his 'Topo-graphia' (1732). He was aware of their Alpine origins, because he wrote that they came from the 'destruction of the moundsains'.

¹⁶ Albrecht von Haller (16.10.1708, Berne – 12.12.1777, Berne) was a Swiss anatomist, physiologist, naturalist, encyclopedist, bibliographer and poet. As an infant prodigy, he had mastered Greek, Hebrew and Latin at the age of 15 when he had already acted as author of numerous translations from Ovid, Horace and Virgil as well written original lyrics, dramas and an epic. He studied medicine in Tübingen and Leiden. His professional activity was started as physician in Berne in 1729, where aside of this, he also studied mathematics and botany for which he produced the basis of his great work on the flora of Switzerland. While there, he conducted many journeys through the Alps. As a result he created in 1729 the poem 'Die Alpen' ('The Alps') that appeared in 1732 as the first edition of his embracing work 'Gedichte' (Poems), demonstrating his appreciation of the mountains. In 1736, Haller was called by King Georg II as professor of medicine, anatomy, botany and surgery to the newly founded University in Göttingen, where he became famous through his work in medicine and botany. He left there in 1753 to return to Berne.

¹⁷ Christian Leopold von Buch (26.04.1774 - 04.03.1853) was a German geologist and paleontologist born in Stolpe an der Oder (now a part of Angermünde, Brandenburg) and is remembered as one of the most important contributors to geology in the first half of the nineteenth century. His scientific interest was devoted to a broad spectrum of geological topics: volcanism, petrology, fossils, stratigraphy and mountain formation. His most remembered accomplishment is the scientific definition of the Jurassic system.

¹⁸ Scott James Hutton (03.06.1726 - 26.03.1797, both in Edinburgh), was an independent scholar and claimed founder of geology, also worked as a physician, and was earlier founder of a chemical factory. He was also active as a multi-year farmer.

how they came.' As primary cause (be aware, existence of earlier ice ages was not known at this time) for the extended glaciers of the Alps, Hutton assumed that the latter had been substantially larger at earlier times. Therefore, it then must have been colder there, which favored an intensified glaciation [because atmospheric temperature decreases with height; this was known at that time]. Damaging of Hutton's concept was that it required a tremendous erosion rate to excavate the suspected high-level plane and to form the valleys from the initially much higher mountains. This would last an excessively long time. Positive is that he is one of the first, who suggested a connection between the phenomenon of erratic boulders and the spacious earlier glaciation.

After Hutton's death, the theologian and mathematician John Playfair continued his work¹⁹. He summarized this work in his book of the year 1802, [142]. His message is the same as that of Hutton. Krüger, [106], quotes him as follows:

'For the moving of large masses of rock, the most powerful engines which nature employs are without doubt the glaciers, those lakes or rivers of ice which are formed in the highest valleys of the Alps, and other mountains of first order. These great masses are in perpetual motion, undermined by the influx of heat from the Earth, and impelled down the declivities on which they rest by their own enormous weight, together with that of the innumerable fragments of rock, which are loaded. These fragments they gradually transport to their utmost boundaries, where a formidable wall ascertains the magnitude, and attests the force, of the great engine by which it was erected', from 'John Playfair, Illustration of the Huttonian theory of the Earth', Edinburgh 1802, p 388ff §348, [142].

Similar observations and interpretations were also made by natural scientists from other countries. In Bavaria the geologist, mineralogist and physicist Mathias von Flurl²⁰ recognized the position-alien character of the erratic boulders in Bavaria. In the year 1809, the Bavarian physician and astronomer Franz von Paula Gruithuisen²¹ devoted some of his research to erratic boulders in the Alps. He knew of the significance of such boulders, and rejected the hypothesis of large floods to be the cause, but kept the transport hypothesis of the apparently unmotivated spreading by water; so, the heritage of the deluge as the principle cause was still behind his thinking. Krüger (2009), [104], cites von Paula Gruithuisen with arguments which today must be interpreted as abstruse geological and hydrological thinking. He, further muses that von Paula Gruithuisen did not seem to know that glaciers do slowly flow down their valleys. They must be immobile like frozen lakes, and a geological force must exist, which pushes the glaciers, including their edgy rocks, upwards to transport the latter large distances. Evidently, such a concept seems today completely unrealistic. The biblical deluge can best serve in this situation as a 'deus ex machina' to resolve this intellectual dilemma.

About at the same time, during the early 19th century, Jean-Pierre Perraudin²² and Siméon Gilliéron²³ came to the conclusion that "The glaciers of our mountain chains [...] had at earlier times a considerably larger extent than today [sic: at Perraudin's and Gilliéron's time]. Our entire valley was up to a high altitude above the Dranse, a river in the Valley de Bagnes, occupied by a single huge glacier that extended far down to Martigny, as the rock boulder evidenced, which were found in the vicinity of this town and which are too big to ever have been transported by water" (translated from Krüger (2009), [106], by KH). Apparently, Perraudin's thesis of 1815 was only based upon positions where erratic boulders were found. This was an indirect recognition that positions of erratic boulders were determined by the earlier glaciation of the valley area, in which they were found.

To summarize:

In the 18th century the contemporary scientists had abandoned the biblical deluge hypothesis in favor of a series of flood events as the causes of the enigmatic positioning of the erratic boulders. Only after the 1740s, first suppositions of the transport of erratic

¹⁹ John Playfair (03.10.1748, Benvie (near Dundee) – 07.20.1819, Burntisland, fife, Scotland) was a Church of Scotland minister, remembered as a scientist and mathematician, and a professor of natural philosophy at the University of Edinburgh. He is best known for his book *Illustrations of the Huttonian Theory of the Earth* (1802), which summarized the work of James Hutton. It was through this book that Hutton's principle of uniformitarianism, first reached a wide audience. Playfair's textbook *Elements of Geometry* made a brief presentation of Euclid's parallel postulate known now as Playfair's axiom. In 1783, he was a co-founder of the Royal Society of Edinburgh. He served as General Secretary to the Society during 1798-1819.

²⁰ Mathias von Flurl (05.02.1756, Straubing – 27.07.1823, Kissingen) is the founder of the Bavarian mineralogy and geology. In 1781, he was called as professor to the University in Munich. From 1788 to 1806, he modernized as factory commissioner the china factory Nymphenburg. 1792 he published the first geological overview of Bavaria in his 'Beschreibung der Gebirge von Baiern und der oberen Pfalz' ('Description of the mountains of Bavaria and the upper Pfalz') including the first geological maps of Bavaria. He thus established the groundwork of geology and mineralogy of Bavaria. Starting in 1822, he alluded the Elector Max IV. Joseph to acquire various collections of minerals and laid the foundation for the 'State of Bavaria's Mineralogical Collection' in Munich. He was also the official inspector of the Bavarian salt-works.

²¹ Franz von Paula Gruithuisen (19.03.1774, Burg Haltenberg am Lech – 21.06.1852, Munich) was a Bavarian physician and astronomer. He taught medical students before becoming a professor of astronomy at the University of Munich in 1826. He believed that the Earth's moon was habitable and was the first to suggest that craters on the Moon were caused by meteorite impacts.

²² Jean-Pierre Perraudin (25.04.1767, Lourtier – 03.01.1858, Lourtier) from the Canton of Valais, Switzerland was first member of the local council of the village Bagnes and later member of the council of the Canton of Valais. He concluded, based on observations that originally much larger glaciers transported erratic boulders and that the ordered scratches at rock walls must have been caused by rocky boulders that were moved and rotated at the glacier beds. When in 1818 the ice dammed lake of the Glacier Giétron erupted, he informed the engineer Ignaz Venetz, who communicated this idea that later was taken up by Jean Charpentier and Louis Agassiz.

²³ Gilliéron Jean-Siméon-Henri (1779-1838), was a deacon in Vevey, at Lake Geneva.

boulders by glaciers appeared, but the principal postulated causes were largely based upon rather unrealistic theoretical concepts, which even were contradictory. The question, how glaciers would move - even the problem whether they would move at all - remained controversial. This remained so in the 19th century. Reflected on from the 21st century, this cannot be a surprise as the climate variations were not yet understood and a continuum mechanical formulation for material behavior had not yet been established.

Systematic Measurement of the Deformation and Motion of Glaciers

Erratic boulders were of help in searching for the deformation and motion of glaciers as a whole. Direct observation of objects on the glacial surface and at certain depths below it, in crevasses and fissures, on the other hand, brought a breakthrough. The detailed observations and measurements were done by James David Forbes²⁴, [44-46], and John Tyndall²⁵ and are expertly described by Tyndall (1860, 1878) [163] and -- in lecture notes 'Eis und Gletscher' ('Ice and Glaciers'), held by Herrmann von Helmholtz, [77], in February 1865 in Frankfurt am Main and Heidelberg. On pp.108, 109 of these notes he writes. 'We have so far compared glaciers according to their manifestation with streams; this similarity is, however, not just an external one; on the contrary, the ice of glaciers moves in fact forward, similar to water in a stream, only slower [...]. Since, namely, the ice at its lower end is incessantly lessened by melting, it would soon have disappeared, if new mass would not continuously move forward from above, which would by snowfalls in the firn fields repeatedly be renewed' (translation KH). Von Helmholtz reports on a number of such convincing situations which we itemize below.

Isolated rock boulders on the surface of a glacier can be used as identifiers of material surface points of glaciers. This observational procedure can be used for any object positioned on the ice and left there, but repeatedly observed for its position.

- In the year 1827 Hugi²⁶ erected a hut on the middle moraine of the Unteraargletscher, to perform there some measurements. The position of this hut was determined by him and by Agassiz 14 years later, in the year 1841; it stood then 4884 feet (~1602m) down-glacier. A somewhat smaller velocity was found by Agassiz for his own hut of the same glacier, [3-5].
- Measuring positions of the surface boulders by theodolites, made by Forbes and Tyndall, from day to day disclosed for the Mer de Glace that during summer the velocities were of the order of 20-35 inch/day (54.7-96.7 cm/day); during winter, they were about half as large; at the ice surface they were larger than at depth, and at the sides they were considerably smaller than in-between'. (von Helmholtz 1865, [77], pp 110-11, translation by (KH))
- The observation of the kind on motion of glaciers allows also an interpretation, in which orientation crevasses must be formed; since, namely, the forward motion of the ice at different positions is not the same, the relative distances of these mass points change with time. Now, because the ice between any two such points cannot be arbitrarily stretched, it will be fractured and form fissures or crevasses' (Helmholtz 1865, [77], p 112, translation KH). Von Helmholtz, quoting Tyndall's conviction that the glacier is hardly able to resist tension but rather ruptures under the influence of such tensions, illustrates this in (Figure 2).
- Both, Tyndall and von Helmholtz, see (Figure 3), beautifully explain how moraines and dust stripes illustrate the deforming motion of the glacier system 'Mer de Glace'.

The three contributing ice-arms are 'Glacier du Géant', 'Glacier de Léchaut' and 'Glacier du Talèfre', which unite as 'Mer de Glace'. Side moraines transporting boulders, fallen from the side flanks of the mountains onto the moving ice, in the three upper glacier arms become in the 'Mer de Glace' interior moraines, except for the outermost two side moraines to the far orientation crevasses must be formed; since, namely, the forward motion of the ice at the far left and far right. The dust stripes in the 'Glacier du Géant' and further down in the 'Mer de Glace' all come from the icefall fed by the 'Col de Géant' (g in panel (b) of (Figure 3)). The material falling down this icefall -- varying from winter to summer -- forms the ice-dust mingling at the base of the icefall, from which the stripes emerge as displayed in (Figure 3), panels (a) and (b). These stripes have in the downstream stretch of the 'Glacier de Géant' a curved appearance across the glacier width that amplifies further down in the 'Mer de Glace'. These

²⁴ James David Forbes (20.04.1809, Edinburgh, Scotland -- 31.12.1868 Bristol, England) was a Scottish physicist and glaciologist, who worked extensively on the conduction of heat and on seismology. He was professor at Edinburgh University from 1833-1859 when he became Principal of the United College of St. Andrews. His principal glaciological research is summarized in 'Travels through the Alps of Savoy' (1843), [46].

²⁵ John Tyndall (02.08.1820, Leighbridge Ireland – 04.12.1893, Haslemere, England) was a prominent 19th-century Irish physicist. His initial scientific fame arose in the 1850s from his study of diamagnetism. Later he made discoveries in the realms of infrared radiation and the physical properties of air. Tyndall also published more than a dozen science books, which brought the state-of-the-art 19th century experimental physics to a wide audience. From 1853 to 1887, he was professor of physics at the Royal Institution of Great Britain in London. As an enthusiastic mountaineer in the Valais Alps of Switzerland he developed a strong interest in glaciology to which he contributed enormously by his books. The Pic Tyndall in the Alps, the Tyndall Mountains in Antarctica, the Tyndall Lake and Glacier in Chile and the Tyndall Moon Crater are named after him.

²⁶ Franz Joseph Hugi (23.01.1791, Grenchen, 25.03.1855, Solothurn, both in Switzerland) was a Swiss geologist and researcher of the Alps. He started as a catholic priest and teacher at the orphanage in Solothurn, Switzerland. There, he founded the Society of Natural Sciences of the Canton Solothurn, which he handed over to the Canton in 1836. Having served in the orphanage and as a teacher in the 'Realschule' (upper level public school) he received in 1833 in the newly opened high school of Solothurn the professorship in physics, but was discharged 1837 because of his transfer to the protestant faith. In the year 1844 he received an honorary doctoral degree from the University of Berne. His research on glaciers is summarized in the scripts 'About the nature of glaciers and winter journey into the polar sea' (1842) and 'The glaciers and erratic boulders' (1843). A detailed evaluation of Hugi's significance as a glaciologist is given in Albert Krehbiel's doctoral dissertation (1902), [104].

bands of dust and the spread of rocks and stones appear as alternating grey and whitish stripes, likely indicating annual repetitions of the ice, as first noted by Forbes. All this is impressive manifestation that velocities at the sides are smaller when compared with the free surface motion in the interior, as so described by Tyndall in the year 1842, [163].

Fig. 20.

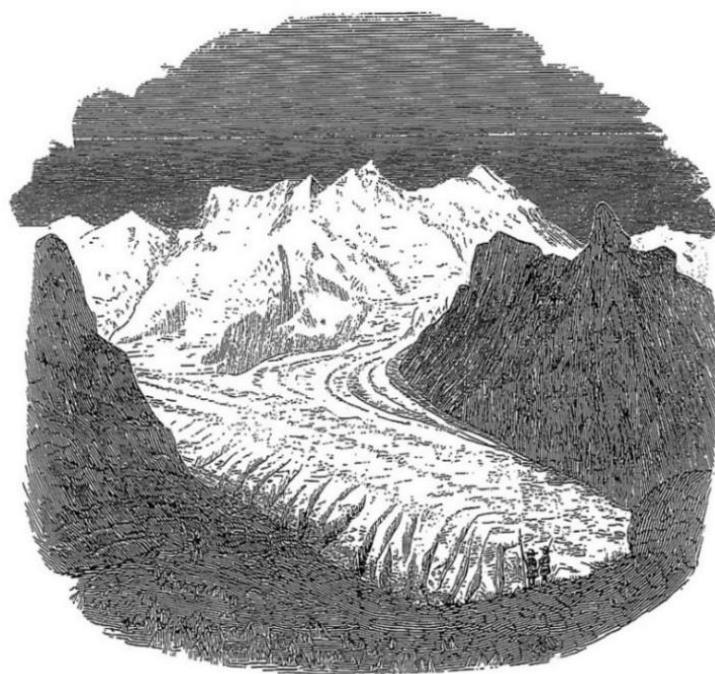


Figure 2: Drawing of the ‘Gornergletscher’ near Zermatt, showing the crevasses in the lowest portion near the snout. The crevasses are approximately perpendicular to the direction of the largest tensile stress, (Helmholtz 1865, [77], p 112).

Fig. 19.

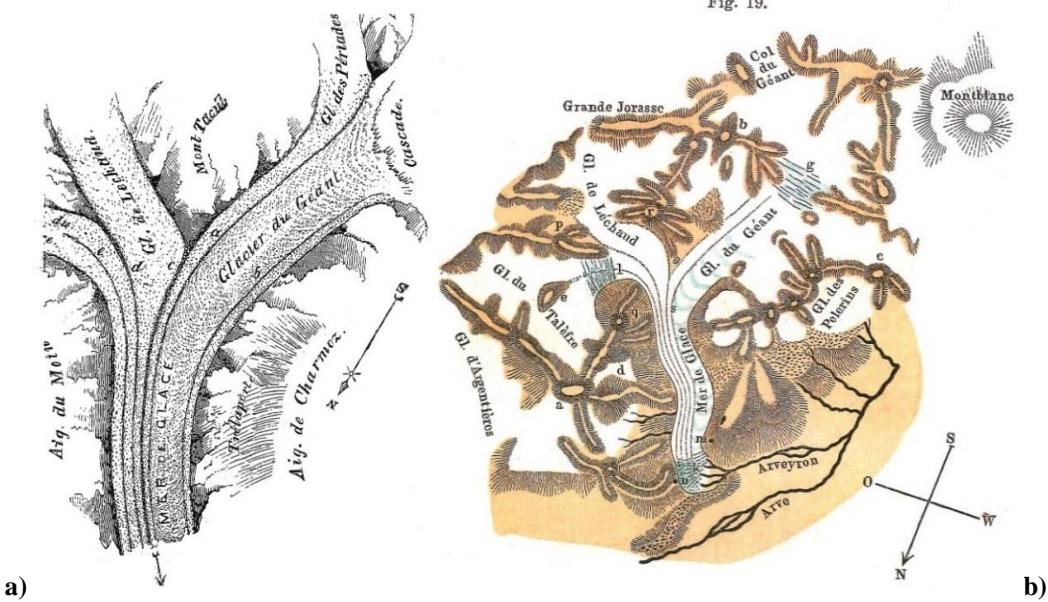


Figure 3: (a) ‘Mer de Glace’, as illustrated by Tyndall (1873, [147] p. 99) and **(b)** by von Helmholtz (1865, [77], p. 110), respectively. The three contributing ice flows are the ‘Glacier de Géant’, Glacier de Léchand’ and ‘Glacier du Talèfre’, which unite as ‘Mer de Glace.

The dust stripes, indicating the displacement distribution of the free surface are in the 'Mer de Glace', limited to the Western part of the glacier; the Eastern part is covered by interior moraines but no curved dusty stripes are seen - at most an onset of spreading

that grows in the down-glacier direction²⁷. Corroboration of this interpretation is given in detail by Tyndall (1873, pp 93-123). He and his collaborators demonstrated this for the ‘Mer de Glace’, the Grindelwald-, Aletsch- and Morteratsch-Gletscher. The method was to insert wooden sticks at several points along an initially straight line into the snowy surface across the glacier (generally 6 to 10 per transect and to determine by theodolite their positions at consecutive times (generally one to a few days apart), see [Figures 4,5].

This allowed evaluating the travelled distance for each wooden stick at consecutive times. This meticulous procedure was the verification of the bold assertion that ‘the ice of a glacier flows slowly, and similarly to a stream of a very viscous substance, as e.g. honey, coal-tar, or a thick tonic pulp’. The ice does not simply slide over the bed, just as a rigid body does, which slides down a slope; it rather bends and displaces itself in itself and, even though it glides over the base of the valley, the parts, which are in contact with the bottom surface and the side walls of the valley, are effectively slowed down, caused by the significant friction. By contrast, the middle of the free surface of the glacier, which is farthest from the bottom and the walls of the valley, move faster. Louis Rendu and James David Forbes were amongst the first to emphasize the similarity of glacier flows with the flow of a fluid substance’ (Helmholtz, 1865, [77], p 116, translation KH).

Thus, by the 1860s the scientists focusing on the physics of the glaciers had acquired convincing knowledge that glaciers were slowly moving and deforming similarly as water in rivers, creeping soil down mountain slopes, hot lava as rivers in volcanic eruptions, honey or polymeric fluid substances. However, a mathematical formulation of such motions as continuous media was not available at that time. Neither was there any mention that inertial terms in Newton’s law in river motions have a sizeable influence, whilst they can safely be ignored in the determination of glacier motions.

To summarize

Systematic measurements of the motion of the ice on the glacier surface at selected points on the surface, at its middle line, close to its side boundaries in fissures and crevasses below the surface were consistently done by Forbes and Tyndall using theodolitic positioning at consecutive days. The evaluation of these measurements led to the interpretation that glacier motion fields are analogous to the motion of terrestrial surface river flows, a fact explicitly spelled out by Rendu in the first half of the 19th century, see [143].

Birth of the Concept of an Ice Age

According to Rowlinson²⁸ (1971), [145] the years from 1840 to 1870 were amongst the most fruitful ones in the history of physics. As for the solution of the problem of the position of the erratic boulders the systematic measurements of the motion and deformation of glaciers primarily done by Agassiz, Forbes and Tyndall were important as was Rendu’s ‘Théorie des Gaciers de la Savoie’ (1840), [143]. Forbes had formulated a model of viscous deformation, which was criticized, because it conflicted with the sliding model, but Forbes successfully showed that the sliding model was in conflict with the (experimental) facts, which he had discovered (Rowlinson 1971, [130]). By 1851 Forbes’ measurements were summarized in his article of 1855 for the eighth edition of the encyclopedia Britannica. Abbreviated as evident, this statement reads

“Each portion of a glacier moves...in a continuous manner...The ice in the middle part of the glacier moves much faster than that near the sides or banks; also the surface moves faster than the bottom.... The variation of velocity (as in a river) is most rapid near the sides.... The glacier, like a stream, has its pools and rapids. Where it is embayed by rocks it accumulates, its declivity decreases, and its velocity at the same time...and the increased temperature of the air favor the motion of the ice.... The velocity does not, however, descend to nothing even in the depth of winter.... [These] circumstances of motion ... appear to be reconcilable with the assumption of what may be called the Viscous or Plastic Theory of glacier motion, and with that alone ... the

²⁷ The difference in these dust distributions is caused by the fact that the ice in the glacier de Léchaud does not fall through an ice fall as at g, as seen in panel (b) of (Figure 3); here, the discharge difference in summer and winter generates annual repetitions of the amount of dust in the Mer de Glace that causes the curved dust stripes.

²⁸ Sir John Shipley Rowlinson (12.05.1926 – 15.08.2018) was a British chemist, who earned his doctorate at the University of Oxford and had research and teaching positions at the University of Wisconsin, the University of Manchester, the Imperial College London and Oxford University, where he retired as professor in 1993. His research was on capillary and cohesion, but he also wrote about the history of science. He was a Fellow of the Royal Society of London and the Academy of Engineering. He routinely climbed the Swiss Alps and the Himalaya.

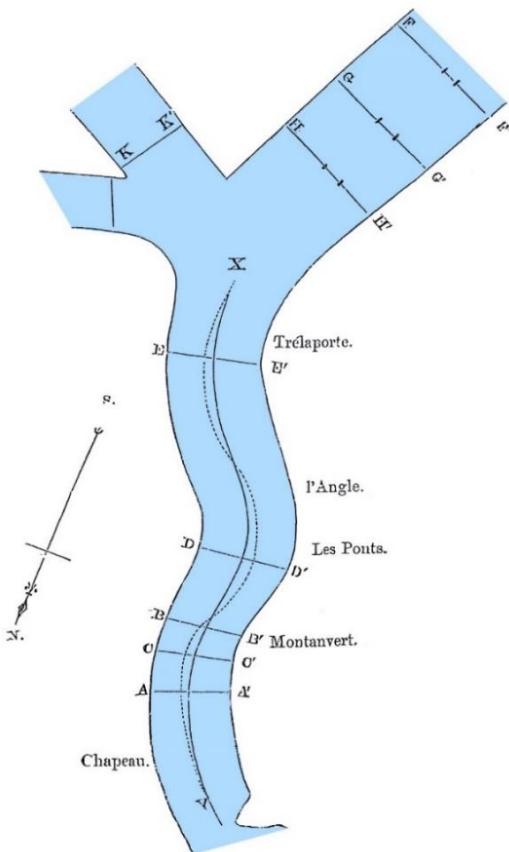


Figure 4: Sketch of the map of the Mer de Glace and its neighboring contributing glaciers, with indication where the cross sectional measurements were made. The middle line (solid) halves the width between the side boundaries. This line is not the location of the fastest surface speed. The line of the fastest surface speed lies closer to the convex side of the middle line (dashed); from Tyndall, with changes [163], pp. 80-91.

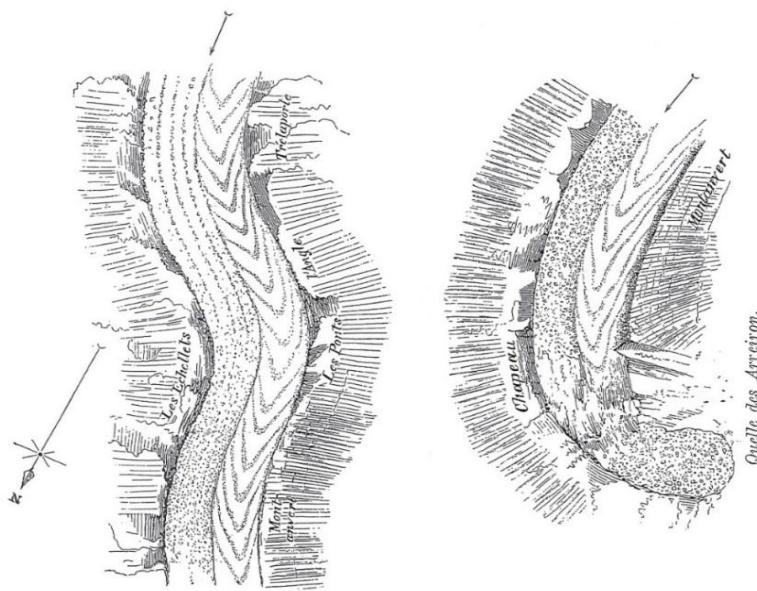


Figure 5: Dirt stripes (left) of the 'Mer de Glace'. Its middle moraine separates the ice of 'Glacier du Géant' from that of the 'Glacier de Léchaud' and Glacier du Talèfre', which shows only traces of moraines; these diffuse to the sides. Tyndall suspects and concludes that the processes in the ice fall produce an annually repeated formation of an ice-dust mingling (via quasi-periodic ice rock slides). (right) End of the dust stripe when the glacier curbs into the snout area. From Tyndall (1860, 1898), [163, 164], differently arranged.

veined or ribbon structure of the ice is the result of internal forces, by which one portion of the ice is dragged past another.... The veined structure is unquestionably the result of the struggle between the rigidity of the ice and the quasi-fluid character of the

motion impressed upon it. That it is so evident not only from the direction of the laminae, but from their becoming distinct exactly in proportion to their nearness to the point where the bruise is necessarily strongest".

Rowlinson's paper is in large parts a detailed historical account of the dispute and disagreement over how the motion of glaciers had to be physically interpreted. "Tyndall's criticism centered on the word 'viscous', which Forbes never defined clearly. He apparently understood it to mean resistance to shear stress, but used the terms viscous, plastic and semi-fluid indifferently... Tyndall interpreted it differently." By viscosity he understands that property of a semi-fluid body, which permits of its being drawn out when subjected to a force of tension. He denied that ice could be permanently stretched without braking and so dismissed Forbes' viscosity as apparent, not real. Instead he proposed his theory of 'fracture and regelation'..." (Rowlinson, 1971, [145]). He describes in detail this somewhat bitter fight that lasted for years; it also bore elements of jealousy, because Forbes feared that Tyndall (the younger of the two) might get the Copley Medal of the Royal Society in 1859. William Thomson 1824-1907 (later (1892) Lord Kelvin), apparently was also involved in this maneuver.

Thermodynamicists of the 19th century played equally a role in the process of creating a realistic physical basis for the motion of glaciers. At that time, Julius Robert Mayer (1814 - 1878) [119] (a physician in Heilbronn, Germany) had formulated the First Law of Thermodynamics - then called the mechanical law of heat - and Rudolf Clausius had introduced the Second Law of Thermodynamics [28, 29], as did Lord Kelvin [161], who introduced in 1848, at the age of 24, the absolute temperature. This was known already earlier, since ice floats on water. So one knew of the expansion of water in the freezing process to ice, and it is, thus, not so surprising that Johann Jakob Scheuchzer and later Jean de Charpentier, [26] and Louis Agassiz, [4, 5] at least for a while assumed that only freezing of intra-glacial water would cause the downward motion of glacier ice.

Nothing of this provides a hint as to the concept of ice ages; neither Agassiz nor de Charpentier nor any other of the glacier scientists of that time proposed the concept of very cold climates in the past. Today the creation of the idea 'ice age' is largely still attributed to Louis Agassiz, but this is erroneous: Surprisingly, it is Johann Wolfgang von Goethe (1749-1832), one of the world's greatest poets, who was equally a respected scientist. Dorothy Cameron (1965), [24], in Vol. 5 of the Journal of Glaciology tells the story. Goethe as a scientist was the first to attribute the transport of erratic blocks to glaciers; he spoke out that an ice sheet covered Northern Germany; furthermore, he is given the credit to first having believed in an ice age. Cameron (1965), [24], states that Louis Agassiz (1837) and Jean de Charpentier (1841) were correct when they assigned to Goethe the credit for being the first to have conceived the concept²⁹ of the ice sheet. There are very few written attributes to this fact. We have only found Reinhard Hederich (1898), [76], being mentioned in a footnote of Alfred Krehbiel's dissertation of (1902), [104]. The comment is: '...neither should Goethe be forgotten, who built himself noteworthy correct views of glaciers and erratic boulders...' Similarly, also Forbes (who was not directly addressing erratic boulders or the concept of ice ages) was well aware of Goethe's scientific significance. On the title page of his 'Travels through the Alpes' he cites Goethe's sentence: 'Sage mir was du an diesen kalten und starren Liebhabereyen gefunden hast' (tell me what you believe to have found in these cold, rigid fondnesses).

The reference to Goethe is also missing in the reviews by Clarke (1987), [27], Walker & Waddington (1988), [171] and Waddington (2010), [170].

Juvenile Struggle with a Creep Law for Ice

As explained in the last section, it was in the 19th century that natural scientists recognized that glaciers are slowly moving ice masses. Helmholtz (1865), [77] and Tyndall (1860, 1878), [163,164], summarized the knowledge as of ~1870; they beautifully describe the slow birth of this new understanding of glacier flows as movements of a very viscous material. For the recognition of this property of deformability a large number of scientists were involved, besides Tyndall, at least Forbes and Rendu ought to be mentioned. At the mid 19-hundreds, further progress in developing a physically based theory for the motion of large ice masses was hampered by two facts,

- The nonexistence of a formal observer-invariant material theory, coupled with the basic physical laws,
- The primitive state of experimental techniques for the determination of the constitutive relations (here primarily for the Cauchy stress tensor).

The second half of the 19th and the early 20th century were needed to develop these two physical, mathematical and engineering-type specialties. As a prelude to the glaciological activities starting shortly before 1950, it is important to note that the material sciences of continuous media -- fluids and solids -- as they developed in the second quarter of the 20th century were active in a nascent field, devoted to the description of the mainly continuous deformations of bodies to external driving elements such as forces, temperature, etc. This new science was coined rheology and became quickly fashionable among chemists, material scientists, applied mathematicians and physicists. The important features in the field of material creep to be solved were how e.g. the stress tensor can be expressed by deformation measures, such as strain and/or rate of strain tensors, temperature, etc. in a materially objective manner, i.e. observer invariant form. Markus Reiner (1886 – 1976), an Austrian-Israeli (civil) engineer, was an early protagonist of the description of the creep behavior of fluids and developed prior to 1929 together with his scientific

²⁹ Cameron writes 'theory' instead of 'concept', but a theory was established only later by Milankovich (1930), [124].

associate Ms. R. Riwlin the Reiner-Riwlin constitutive relation for fluid creep. Expressed in a nutshell: Let \mathbf{t} be the stress tensor and \mathbf{t}' its deviator and \mathbf{D} the strain rate tensor³⁰. These are symmetric 3x3-tensors, and in a volume-preserving (i.e. incompressible) material, \mathbf{D} is a deviator (its trace or first invariant vanishes). So, a possible constitutive relation must be expressible as $\mathbf{D} = \mathbf{f}(\mathbf{t}', T)$, where the function \mathbf{f} , when evaluated in terms of \mathbf{t}' and T , must equally be a deviator. The explicit form of \mathbf{f} , satisfying these requirements can be constructed with relatively simple methods of 3x3-matrix algebra and is given by

$$\mathbf{D} = \psi_1 \mathbf{t}' + \psi_2 [(\mathbf{t}')^2 - \frac{2}{3} II_{\mathbf{t}'} \mathbf{1}],$$

in which both sides of this relation are deviators, $\psi_{1,2}$ are scalars given by

$$\psi_{1,2} = Fcts(II_{\mathbf{t}'}, III_{\mathbf{t}'}, T), II_{\mathbf{t}'} = \frac{1}{2} \text{tr}(\mathbf{t}')^2, III_{\mathbf{t}'} = \det(\mathbf{t}')$$

and $II_{\mathbf{t}'}$ and $III_{\mathbf{t}'}$ are the second and third invariants of \mathbf{t}' (while the first invariant $I_{\mathbf{t}'}$ vanishes). The above expressions are in conformity with isotropy. We ask the reader to accept the above polynomial expression as the most general expression for $\mathbf{f}(\mathbf{t}', T)$ under the stated conditions. In ensuing developments, the term involving ψ_1 will be called the collinear (or affine) term to \mathbf{D} for obvious reasons. The second term, involving ψ_2 , may then be called the quadratic term.

The utmost majority of scientific works in glaciology has been done for the case that $\psi_2 = 0$ and that ψ_1 does not depend on $III_{\mathbf{t}'}$. Investigations have been done in which $\mathbf{D} = \psi_1(II_{\mathbf{t}'}, III_{\mathbf{t}'}, T)\mathbf{t}'$, but it will be shown that for polycrystalline isotropic ice an extended dependence also involving $III_{\mathbf{t}'}$ can consistently only be introduced when the quadratic term involving ψ_2 is included.

The remainder of this historical review will be devoted to the test of adequacy of the Reiner-Riwlin fluid as a constitutive model for the creep of polycrystalline isotropic ice. Naturally, this historical episode only dates back to the last 60-70 years with important contributions done since ~1980. We do this in this historical article against the recommendation of two reviewers and the handling Scientific Editor of the *J. Glaciology*. Referee 1 of that version of this paper stated

“There is an important borderline between <telling the history> and <advocating a scientific conclusion>, and I [sic: the referee] believe that the paper tends to cross the border from history toward the side of expressing a scientific opinion. I view this to be a point where the paper is no longer about history but is about new science being advocated...”

We as authors disagree with this request that the two parts ought to be separated. Quite contrary: Historical facts ought to be used for the consequences, which they imply. Trivially, likely all facts of our manuscript have happened in the past (with the credits given to others) and are therefore part of the history.

Applied glaciologists have almost exclusively employed the Reiner-Riwlin fluid model in its reduced form

$$\mathbf{D} = \psi_1(II_{\mathbf{t}'}, T)\mathbf{t}',$$

in which ψ_1 is a power law expression of $II_{\mathbf{t}'}$, multiplied with a temperature dependent rate factor. For temperatures distant from the melting point this dependence is expressed by an Arrhenius-type relation, and close to the melting point generally constructed by optimizing a functional ansatz to observed temperature data. Polynomial fits to temperature data have also been done and show generally a better agreement with data points. However, when comparing optimally determined expressions from one experimental site to another (or from one author group to another) agreement is, more often than not, disastrous. Smith and Morland (1981), [130] have documented this. Despite this demonstration, practically all glaciologists not involved in the (experimental or theoretical) determination of the ‘flow law for ice’, apply the power law unquestioned as if it were the commandment of a glaciological deity. Nye’s paper of (1952), [135], may well serve as the trigger of such attitudes. In that paper, two basic assumptions for the strain-rate--stress relation were that (i) this relation did not depend upon the third invariant $III_{\mathbf{t}'}$ and (ii) the strain rate deviator is collinear to the stress deviator. Given the prerequisite that Orowan was a leading scientist at the Cavendish Laboratory in Cambridge at that time (see his paper of (1949), [136]) involved with plastic modeling of materials, and Nye and Glen were junior members there, it is but natural that Glen came up with a power law. Let us now illustrate, how the last 30-40 years were vital for our thesis that polycrystalline ice ought to be modeled as a full Reiner-Riwlin fluid or even as a more general fluid.

The scene began in the 1950s with the recognition that creep of ice in large ice masses must be treated as an isotropic nonlinear viscous or plastic body. Creep tests on polycrystalline ice were done first by Perutz (1950), [141]; Glen (1952, '53, '54, '55, '58, '74), [55-61] and Steinemann (1954, '56, '58), [157-159], followed by Gold (1958), [61-63], Mellor & Smith (1967), [120], Mellor & Testa (1969a,b), [121, 122], Jacka (1984), [97], with reviews given by Kuo (1972), [108], Shumskiy (1974), [154] and Mellor (1979), [123]. The commonly used stress systems are tension, compression and simple shear, although combined states of stress had also been looked at. In addition, the attitude of material scientists were, that experimental results obtained with such simple stress systems could be extrapolated to 2- and 3-dimensional stress systems.

³⁰ All bold faced quantities are second rank tensors or 3x3-matrices: $\mathbf{D}, \mathbf{t}, \mathbf{t}', \mathbf{f}$.

Typical creep curves for polycrystalline ice under simple stress states (shear, tension/compression) are as shown in (Figure 6) with primary, secondary and tertiary creep, where

- Primary creep is decelerating,
- Secondary creep is steady
- and tertiary creep is accelerating, merging into another steady state or leading to rupture.

Experiments by Steinemann (1958), [153], see (Figure 6a,b) show how creep curves look like in these ranges and (Figure 7a,b) shows analogous curves in doubly logarithmic representation. Since in these latter plots $\ln(\dot{\epsilon})$, plotted

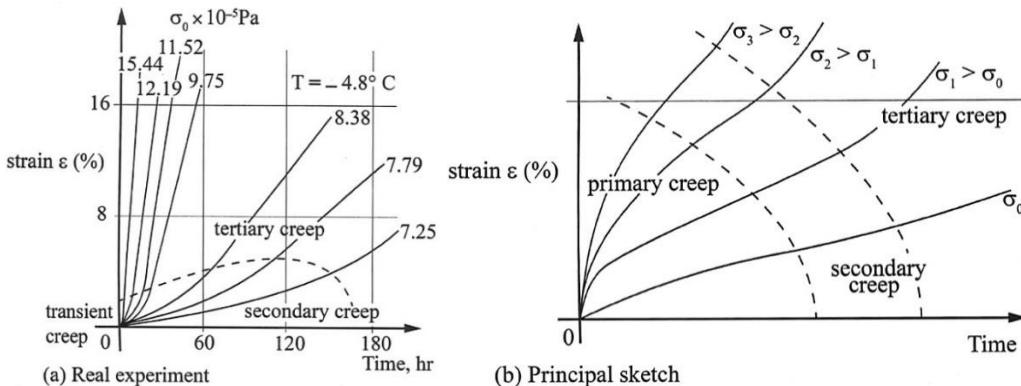


Figure 6: (a) Creep curves for polycrystalline ice at -4.8°C under various stresses at first and secondary creep, from Steinemann (1958). (b) Principal sketch, showing first, secondary and tertiary creep, principal sketch, from Hutter (2018), [90].

against $\ln(\sigma)$, are not straight lines, these results provide corroboration for the fact that the power law behavior with constant exponent cannot describe the appropriate strain-rate relation³¹. Steinemann writes

$$\dot{\epsilon} = \text{sgn}(\sigma)f(T)|\sigma|^{n(\sigma,T)}, f(T) = \exp\left(-\frac{Q}{kT}\right) \quad (2)$$

Note, we write for the one-dimensional strain rate now $\dot{\epsilon}$ (in lieu of $|\mathbf{D}|$) and σ (in lieu of $|\mathbf{t}'|$) as this is the notation employed by Steinemann and Glen and used by engineers. Steinemann and Glen approximate these expressions as

$$\dot{\epsilon} = \text{sgn}(\sigma)f(T)(\sigma^2)^{(n-1)/2}\sigma, n = \text{const.}, \quad (3)$$

requesting that this law is restricted to a finite stress range $\sigma \in [\sigma_l, \sigma_u]$, with different constant values of n in different stress ranges of lower σ_l and upper σ_u .

The above representations (2) and (3) are restricted to uniaxial normal stress and simple shear, but the structure of these formulas was wished to be also applicable to 3-dimensional stress states. For density preserving materials one may write an objective strain-rate stress relation for polycrystalline isotropic ice as a Reiner-Riwlin fluid; i.e., if \mathbf{t} is the Cauchy stress and \mathbf{t}' its deviator, the stretching-stress relation reads

$$\mathbf{D} = \psi_1 \mathbf{t}' + \psi_2 [(\mathbf{t}')^2 - \frac{2}{3} \mathbf{I} \mathbf{I} \mathbf{t}' \mathbf{1}], \quad (4)$$

in which both sides of this relation are deviators and

$$\psi_{1,2} = \text{Fcts}(\mathbf{I} \mathbf{I} \mathbf{t}', \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{t}', T), \mathbf{I} \mathbf{I} \mathbf{t}' = \frac{1}{2} \text{tr}(\mathbf{t}')^2, \mathbf{I} \mathbf{I} \mathbf{I} \mathbf{t}' = \text{det}(\mathbf{t}'). \quad (5)$$

$\mathbf{I} \mathbf{I} \mathbf{t}'$ & $\mathbf{I} \mathbf{I} \mathbf{I} \mathbf{t}'$ are second and third invariants of \mathbf{t}' . This relation was made plausible above when we gave a wordy explanation of the Reiner-Riwlin³² constitutive relation. Glacial theorists must in principle have known the relation in the 1950s. John Nye (1952, 1953), [121, 122], took this to postulate that,

- (i) the third invariant, $\mathbf{I} \mathbf{I} \mathbf{I} \mathbf{t}'$, has no effect on the material response and,
- (ii) the strain rate tensor and stress deviator are co-axial.

These postulates require that $\psi_2 = 0$ (see, equation (4)) and

$$\mathbf{D} = \psi(\mathbf{I} \mathbf{I} \mathbf{t}', T) \mathbf{t}', \psi := \psi_1, \rightarrow \mathbf{I} \mathbf{I} \mathbf{D} = \psi^2(\mathbf{I} \mathbf{I} \mathbf{t}', T) \mathbf{I} \mathbf{I} \mathbf{t}', \quad (6)$$

³¹ We take here the position that the reader knows that a graph, in which $\log(y)$ is plotted against $\log(x)$ as a straight line corresponds to a power law of y against x .

³² This fluid is called after M. Reiner and R. Riwlin, but is by far mostly referred to as 'Reiner-Riwlin fluid'. Before 1929, Reiner had worked on a number of projects, now considered as rheology, as said by Scott-Blair (1976), [150]!, 'he developed the 'Reiner-Riwlin equation' – his colleague, Miss R. Riwlin was killed in a road accident. Her nephew, Roland S. Rivlin (now spelled with a 'V') is a famous [British]-American rheologist'. The Reiner-Riwlin equation does not reproduce primary, secondary and tertiary creep for that purpose, the constitutive stress must also depend on higher order Rivlin-Ericksen tensors (see e.g. Man, et al. (1985, '87, '92, 2010), [115-119], and Hutter (2019), [90]).

valid as the general constitutive relation for density preserving polycrystalline ice.

As his example, Steinemann chose a combined unilateral compression plus shear test as follows

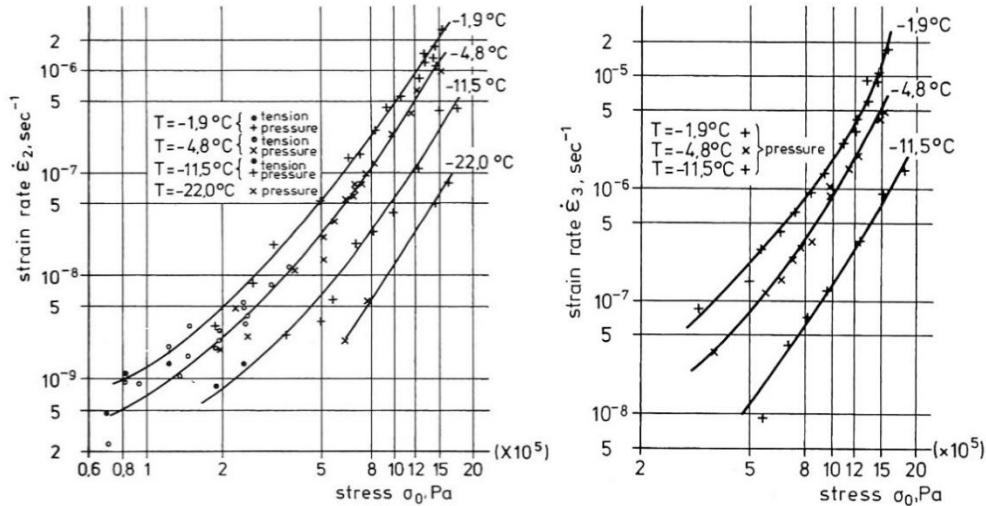


Figure 7 (left) Strain rate during secondary creep for polycrystalline ice as a function of stress. **(right)** Strain rate for tertiary creep of polycrystalline ice as a function of stress, from Steinemann (1958), [158] as drawn by Hutter (1982), [90].

$$\mathbf{t} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & -p & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \mathbf{t}' = \begin{pmatrix} p/3 & \tau & 0 \\ \tau & -2p/3 & 0 \\ 0 & 0 & p/3 \end{pmatrix} \quad II_{\mathbf{t}'} = \frac{p^2}{3} + \tau^2, \quad III_{\mathbf{t}'} = -\frac{p}{3} \left(\frac{2}{9} p^2 + \tau^2 \right), \quad (7)$$

which, under ideal experimental performance of a volume preserving isotropic material, yields

$$\mathbf{D} = \frac{1}{2} \begin{pmatrix} \dot{\varepsilon} & \dot{\gamma} & 0 \\ \dot{\gamma} & -2\dot{\varepsilon} & 0 \\ 0 & 0 & \dot{\varepsilon} \end{pmatrix}, \quad II_{\mathbf{D}} = \frac{1}{2} \left(3\dot{\varepsilon}^2 + \frac{\dot{\gamma}^2}{2} \right), \quad (8)$$

$$III_{\mathbf{D}} = -\frac{\dot{\varepsilon}}{8} (2\dot{\varepsilon}^2 + \dot{\gamma}^2).$$

Steinemann did perform such creep tests and measured $\dot{\gamma}$, but not $\dot{\varepsilon}$; so, his experiments are insufficient to determine the functions ψ_1 and ψ_2 in (4) by experiment. However, one can test whether Nye's postulates, leading to (6) are reasonable. His co-axiality postulate between \mathbf{t}' and \mathbf{D} requires that

$$\frac{\tau}{\dot{\gamma}/2} = \frac{2p/3}{\dot{\varepsilon}} \rightarrow \dot{\varepsilon} = \frac{p\dot{\gamma}}{3\tau}, \quad (9)$$

for which case, according to (8)₂, we obtain

$$II_{\mathbf{D}} = \frac{\dot{\gamma}^2}{2} \left(\frac{1}{2} + \frac{p^2}{3\tau^2} \right), \quad II_{\mathbf{D}} = \psi^2 (II_{\mathbf{t}'}, T) II_{\mathbf{t}'} * = * f \left(\frac{p^2}{3} + \tau^2 \right) = f(II_{\mathbf{t}'}), \quad (10), \quad (11)$$

in which (11) follows by use of (6). Moreover, at * = *, we have assumed isothermal conditions. If one now plots the values of $II_{\mathbf{D}}$ against $(p^2/3 + \tau^2)$ one should obtain a single curve, independent of whether compression is acting together with shear or not. Glen (1958) [59] performed this comparison, which is illustrated in (Figure 8); the figure caption reports on details. It is clearly seen that the experimental points cannot be regarded to approximate a single curve. It follows, co-axiality of the constitutive part of the stress, \mathbf{t}' and the strain-rate tensor \mathbf{D} cannot be maintained as a postulate of a creep law in combined stress states of isotropic polycrystalline ice. We shall below have the occasion to see further arguments, which support this evidence. For multi-axial stress-strain-rate relations there is no experimental support for Nye's co-axiality assumption between \mathbf{t}' and \mathbf{D} . It is interesting that this more than 60 year old finding has not yet seriously been taken up in applied glaciology.

The combined compression plus shear test with the expressions (7) and (8) provides a possibility to study the relevance of the third invariant(s) as independent constitutive variables. Equations (7)₄ and (8)₃ imply

- in simple shear with $p = 0$ and $\dot{\varepsilon} = 0$, we have $III_{\mathbf{t}'} = 0$ and $III_{\mathbf{D}} = 0$,
- in uni-axial compression/tension without shear, $\tau = 0$ and $\dot{\gamma} = 0$, we have $III_{\mathbf{t}'} = -\frac{2p^3}{27}$, and $III_{\mathbf{D}} = -(\dot{\varepsilon})^3/4$.

It follows that independence of the constitutive relation for the Cauchy stress deviator from the third invariant implies that creep laws obtained for compression/extension and simple shear, respectively, must be brought to coincidence. If this coincidence

cannot be verified by experiments to within acceptable errors, then a dependence of the stress strain-rate relation on the third invariant is likely.

To summarize

Multi-axial deformation tests (shear plus compression) show that isotropic ice does not satisfy Nye's assumption that the creep law for the Cauchy stress is independent of the third invariant, see (Figure 8).

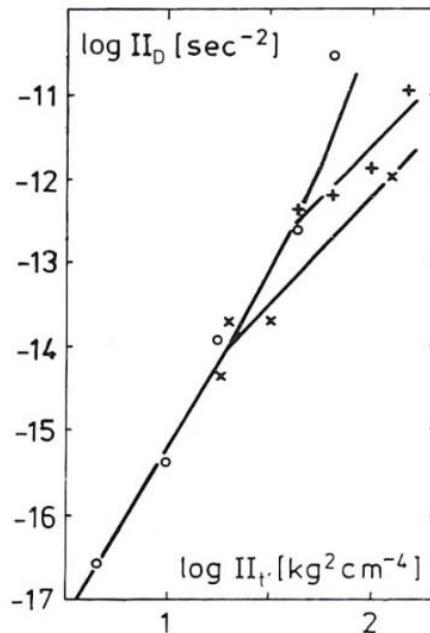


Figure 8: A graph of $\log(II_D)$ against $\log(II_{t'})$ using the data of Steinemann's tests under combined shear and compression. If the assumption (i) suggested by Nye were true, all the points should lie on one curve. The symbols used for points have the following meaning: \circ points, derived from tests in simple shear, \times points derived from tests for a shear stress of 4.20 kg cm^{-2} with superimposed compressive stresses from 0 to 18 kg cm^{-2} , $+$ points derived from tests under a shear stress of 6.59 kg cm^{-2} with superimposed compressive stresses from 0 to 18 kg cm^{-2} , $1 \text{ kg cm}^{-2} = 9.81 \text{ Pa}$, from Glen (1955), [58].

Questionable Consolidation of the Co-axiality Hypothesis in the 1960s to 1980s

- The period after the formulation of the power law by Glen and Steinemann was not characterized by searching for, and extension of, the flow law to a general constitutive relation of a nonlinear fluid. Experimentalists rather seemed to be trapped in the simplicity and beauty of Nye's two postulates, [135]. The power law was replaced by functions better matching the experiments (e.g. $\sinh(\bullet)$ to the n^{th} power, see Barnes, et al. (1971), [10]). Another disadvantage of these 'generalized flow laws' was the problem of infinite viscosity (or vanishing fluidity) at zero stress. More precisely, the constitutive part of the stress would infinitely quickly change at stress initiation. This is unphysical and can be removed by replacing the power law for viscosity and fluidity, respectively, by e.g. polynomials, which include a constant term that mimics the finite viscosity at zero stress. These polynomials also allow a better fit of the coefficients of the creep law with experimental data, as demonstrated by Smith and Morland (1981), [155]. These authors work with effective strain-rates and stresses defined by

$$\dot{\varepsilon}_{\text{eff}} = II_D^{1/2}, \tau_{\text{eff}} = II_{t'}^{1/2} \quad (12)$$

and write the constitutive law (6) for isothermal processes as $II_D = II_{t'} \psi^2(II_{t'})$. This can then be written as

$$\psi((\tau_{\text{eff}})^2) := \frac{\dot{\varepsilon}_{\text{eff}}}{\dot{\varepsilon}_0} \frac{\tau_0}{\tau_{\text{eff}}} = \{c_0 + F_{\text{Name}}((\frac{\tau_{\text{eff}}}{\tau_0})^2)\} \quad (13)$$

where $c_0 \neq 0$ accounts for a finite viscosity at zero stress, whereas $\dot{\varepsilon}_0 = 200 \text{ a}^{-1}$, $\tau_0 = 5 \times 10^5 \text{ Pa}$ are reference scales and

$$F_{\text{Glen}}((\frac{\tau_{\text{eff}}}{\tau_0})^2) = \frac{3}{2}k (3(\frac{\tau_{\text{eff}}}{\tau_0})^2)^{(n-1)/2}, \quad \text{Glen (1953)}, \quad (14)$$

$$F_{\text{SM}}((\frac{\tau_{\text{eff}}}{\tau_0})^2) = \sum_{j=0}^J c_j (\frac{\tau_{\text{eff}}}{\tau_0})^{2j}, \quad (J = 2 \text{ or } J = 3), \quad \text{Smith \& Morland (1981)}.$$

The coefficients c_j for data of a number of experiments studied by Smith & Morland (1981) are determined by using an objective function minimizing the sum of the distances squared of all data points from the mathematical expressions (10) and (11), respectively.

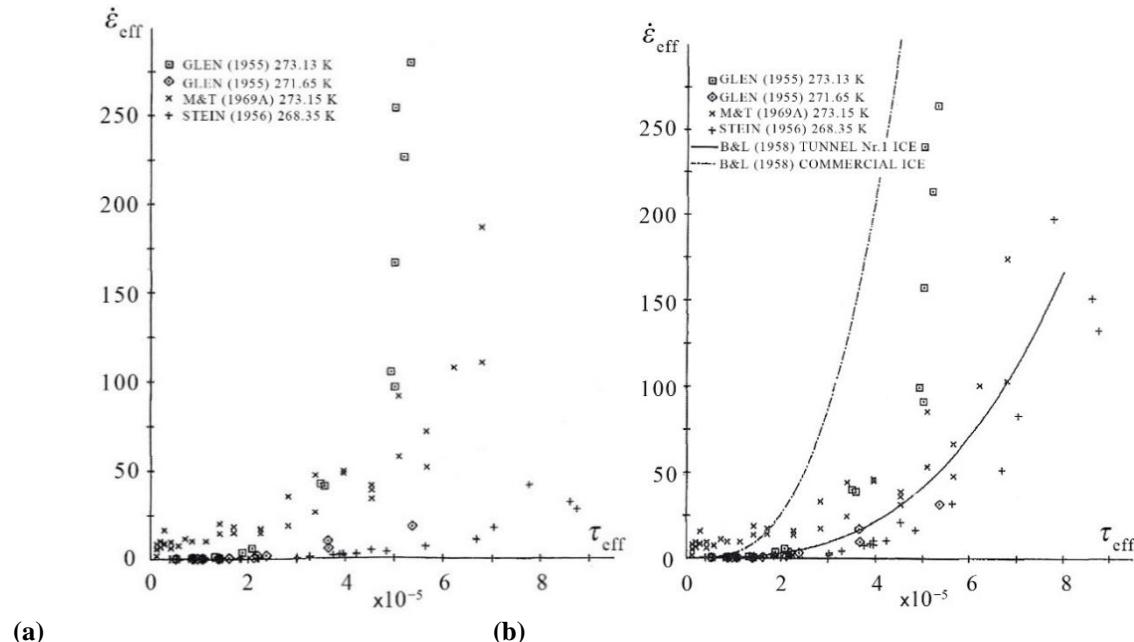


Figure 9: (a) Various strain-rate stress data from Glen, [59], Mellor & Testa (1969a,b), [122], and Steinemann 1956) [159], identified in the inset. (b) Same as in panel (a), but now normalized to the same temperature at 273.13 K. The panel also shows two polynomial laws due to Butkovich & Landauer (1960), [19]. The strain-rate unit is given in a^{-1} and the stress in 10^5 Pa, Redrawn from Smith & Morland (1981), [155].

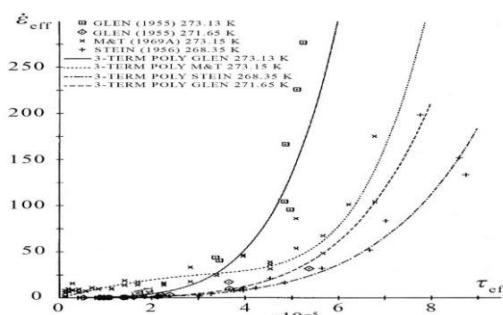


Figure 10: Same as (Figure 9). The separate data sets are correlated with 3-term polynomials ($J = 2$) with polynomials as given in (14). The strain-rate unit is given in a^{-1} and the stress in 10^5 Pa. The experimental points and the construction of the parameterized polynomial are as given in the inset. Redrawn from Smith & Morland (1981), [155].

(Figure 9) collects measured $\dot{\epsilon}_{eff}(\tau_{eff})$ -data points from experiments performed by Glen (1953), [56], Mellor & Testa (1969a,b) [121, 122], and Steinemann (1956), [158]. Panel (a) suggests a tremendous spread of the data; those obtained by different experimenters or from different machines hint at a separate coherent monotonic growth of the $\dot{\epsilon}_{eff}(\tau_{eff})$ -functions, but obviously also point at data incoherence from one experimental site to another one. This interpretation is supported by the two $\dot{\epsilon}_{eff}(\tau_{eff})$ -laws presented by Butkovich & Landauer (1960), [19] from their own experiments. They corroborate the accelerated monotonic growth but global inconsistency with the data of Glen, Steinemann and Mellor & Testa. The coherence of the 3-term polynomial representations for the data of separate experimentalists is demonstrated in (Figure 10).

To summarize:

- Comparison of creep data of polycrystalline isotropic ice with co-axial strain-rate stress relations and independence of these relations of the third invariants, showed that the accelerated monotonic increase of $\dot{\epsilon}_{eff}$ with τ_{eff} is a common feature of all these creep laws.

- The power law description of the $\dot{\varepsilon}_{eff}(\tau_{eff})$ -relation is generally less accurate than polynomials, provided the number of terms in the polynomials, is small (J = 2 in our case).
- The rate factor a(T) of the constitutive relation $\mathbf{D} = \psi(II_{t'}, T) \mathbf{t'}$ varies between different experimental sites.

The above findings suggest that in addition to the temperature dependence at least a further variable, of which the cause may be physical, instrumental or experimental, must also influence the strain-rate stress relation, viz.,

$$\mathbf{D} = E(?) a(T) \psi(II_{t'}) \mathbf{t'}. \quad (15)$$

Many experiments are generally made with grains of pure ice mingled with pure water and then frozen, the rate factor E(?) in the last formula remains presently a mystery, because it does not even tell the variable. Incidentally, eq. (15), also states that the assumption of the thermo-rheological simplicity assumption of the constitutive relation for the stress tensor is likely invalid, see Morland & Lee (1960), [127]. We conclude:

- For simple shear and uniaxial compression Nye's postulates give exactly the same class of constitutive functions, but if the ψ -functions determined by the experiments differ from one another; this difference is an indication that III_D must play a role.
- Steinemann performed this kind of combined experiment, measured $\dot{\gamma}$, but not $\dot{\varepsilon}$. However, this situation still allows at least to check whether the co-axiality postulate between \mathbf{D} and $\mathbf{t'}$ is satisfied. It follows from the definition of the second invariant of \mathbf{D} and the co-axiality assumption of \mathbf{D} ,

$$II_D = \frac{\dot{\gamma}^2}{2} \left(\frac{1}{2} + \frac{p^2}{3\tau^2} \right).$$

On the other hand, equation (11) can for isothermal processes be written as

$$II_D = \psi^2(II_{t'}, T) II_{t'} = f \left(\frac{p^2}{3} + \tau^2 \right) = f(II_{t'}).$$

Glen constructed (Figure 8) more than 60 years ago and demonstrated that Nye's postulate ought to be abandoned.

It is time to do this!

Recent climate relevant research that employs constitutive relations of polycrystalline isotropic ice as a viscous material generally assumes co-axiality of the strain-rate tensor and the Cauchy stress deviator that hardly deviates from the simple power law of the Glen-Steinemann type. This essentially remained so, even though facts to the contrary are well known to the experimental specialists. There is a necessity for a more general form of the constitutive relation with non-coaxiality of the strain rate tensor and the Cauchy stress deviator ($\mathbf{t'}$). The simplest such constitutive relations are the Reiner-Riwlin fluids, which for density preserving fluids have one of the forms

$$\bar{\mathbf{D}} = \psi_1 \mathbf{t'} + \psi_2 [(\mathbf{t'})^2 - \frac{2}{3} II_{t'} \mathbf{I}], \mathbf{t'} = \phi_1 \bar{\mathbf{D}} + \phi_2 [(\bar{\mathbf{D}})^2 - \frac{2}{3} II_D \mathbf{I}]. \quad (16)$$

In these formulae, different from eq. (4), the stretching tensor \mathbf{D} has been written as $\mathbf{D} = a(T) \bar{\mathbf{D}}$, where $a(T)$ is a temperature dependent rate factor. Moreover,

$$\psi_{1,2} = \psi_{1,2}(II_{t'}, III_{t'}, T), \phi_{1,2} = \phi_{1,2}(\bar{II}_D, \bar{III}_D, T);$$

$$\bar{II}_D = II_D, \bar{III}_D = III_D \text{ and } \mathbf{t'} = \mathbf{t} - p\mathbf{I},$$

and p is the pressure.

Morland and Staroszczyk (2019) [127] – whom we follow here - have performed a Gedanken experiment; it is a combination of compression and shear deformation, and it serves as a suggestion to catch the interest of experimentalists. So, let \mathbf{X} and \mathbf{x} be position vectors of ice particles in the reference and present configurations, respectively; then, compressing or extending and shear deformations can be described as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \kappa \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1/\lambda_1 & 0 & -\kappa/(\lambda_1 \lambda_3) \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}. \quad (17)$$

Here, λ_a ($a = 1,2,3$) are normal stretchings and κ is the shear strain rate in the xz-direction. With these relations it is straightforward to derive for the velocity \mathbf{v}

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \dot{\lambda}_1/\lambda_1 & 0 & (\dot{\kappa} \lambda_1 - \kappa \dot{\lambda}_1)/(\lambda_1 \lambda_3) \\ 0 & \dot{\lambda}_2/\lambda_2 & 0 \\ 0 & 0 & \dot{\lambda}_3/\lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

$$\mathbf{L} := \text{grad } \mathbf{v} = \begin{pmatrix} \dot{\lambda}_1/\lambda_1 & 0 & (\kappa \lambda_1 - \kappa \dot{\lambda}_1)/(\lambda_1 \lambda_3) \\ 0 & \dot{\lambda}_2/\lambda_2 & 0 \\ 0 & 0 & \dot{\lambda}_3/\lambda_3 \end{pmatrix}, \quad (18)$$

from which

$$\bar{\mathbf{D}} = \begin{pmatrix} \bar{D}_{11} & 0 & \dot{\gamma} \\ 0 & \bar{D}_{22} & 0 \\ \dot{\gamma} & 0 & -\dot{\varepsilon} \end{pmatrix} \bar{\mathbf{D}}^2 = \begin{pmatrix} \bar{D}_{11}^2 + \dot{\gamma}^2 & 0 & \dot{\gamma} \bar{D}_{11} - \dot{\gamma} \dot{\varepsilon} \\ 0 & \bar{D}_{22}^2 & 0 \\ \dot{\gamma} \bar{D}_{11} - \dot{\gamma} \dot{\varepsilon} & 0 & \dot{\varepsilon}^2 + \dot{\gamma}^2 \end{pmatrix}, \quad (19)$$

$$\bar{D}_{11} = \frac{\dot{\lambda}_1}{\lambda_1}, \bar{D}_{22} = \frac{\dot{\lambda}_2}{\lambda_2}, \bar{D}_{33} = \frac{\dot{\lambda}_3}{\lambda_3}, \bar{D}_{31} = \bar{D}_{13} := \dot{\gamma} = \frac{\kappa - \kappa \bar{D}_{11}}{2\lambda_3} \quad (20)$$

and

$$\begin{aligned} \bar{I}_{\mathbf{D}} &= \bar{D}_{11} + \bar{D}_{22} - \dot{\varepsilon} = 0, \\ \bar{I}_{\mathbf{D}} &= \frac{1}{2} \text{tr}(\bar{\mathbf{D}}^2) = \dot{\gamma}^2 + \frac{1}{2} (\bar{D}_{11}^2 + \bar{D}_{22}^2 + \dot{\varepsilon}^2), \quad (21) \\ \bar{I}_{\mathbf{D}} &= -\bar{D}_{22}(\bar{D}_{11} \dot{\varepsilon} + \dot{\gamma}^2). \end{aligned}$$

Li & Jacka (1996), [121], Warner et. al. (1999), [172], Treverrow, et al. (2012), [162] and Budd, et al. (2013), [17], conducted experiments that fit the above conditions and performed these with the assumption

$$\bar{D}_{11} = 0 \rightarrow \bar{D}_{22} = \dot{\varepsilon}, \bar{I}_{\mathbf{D}} = \dot{\gamma}^2 + \dot{\varepsilon}^2 \quad (22)$$

$$\begin{aligned} \mathbf{t} &= \begin{pmatrix} \sigma_x & 0 & \tau \\ 0 & \sigma_y & 0 \\ \tau & 0 & -\sigma \end{pmatrix}, p = -\frac{1}{3}(\sigma_x + \sigma_y - \sigma), \\ \mathbf{t}' &= \frac{1}{3} \begin{pmatrix} 2\sigma_x - \sigma_y + \sigma & 0 & 3\tau \\ 0 & 2\sigma_y - \sigma_x + \sigma & 0 \\ 3\tau & 0 & -2\sigma - \sigma_x - \sigma_y \end{pmatrix} \quad (23) \end{aligned}$$

and, upon using (16)₂, we may deduce

$$\begin{aligned} \mathbf{t}' &= \phi_1 \begin{pmatrix} 0 & 0 & \dot{\gamma} \\ 0 & \dot{\varepsilon} & 0 \\ \dot{\gamma} & 0 & -\dot{\varepsilon} \end{pmatrix} + \phi_2 \left[\begin{pmatrix} \dot{\gamma}^2 & 0 & -\dot{\gamma} \dot{\varepsilon} \\ 0 & \dot{\varepsilon}^2 & 0 \\ -\dot{\gamma} \dot{\varepsilon} & 0 & \dot{\varepsilon}^2 + \dot{\gamma}^2 \end{pmatrix} - \frac{2}{3}(\dot{\varepsilon}^2 + \dot{\gamma}^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] \\ &= \phi_1 \begin{pmatrix} 0 & 0 & \dot{\gamma} \\ 0 & \dot{\varepsilon} & 0 \\ \dot{\gamma} & 0 & -\dot{\varepsilon} \end{pmatrix} + \phi_2 \begin{pmatrix} \frac{1}{3}(-2\dot{\varepsilon}^2 + \dot{\gamma}^2) & 0 & -\dot{\gamma} \dot{\varepsilon} \\ 0 & \frac{1}{3}(\dot{\varepsilon}^2 - 2\dot{\gamma}^2) & 0 \\ -\dot{\gamma} \dot{\varepsilon} & 0 & \frac{1}{3}(\dot{\varepsilon}^2 + \dot{\gamma}^2) \end{pmatrix}, \quad (24) \end{aligned}$$

in which ϕ_1 and ϕ_2 are functions of $\bar{I}_{\mathbf{D}}$ and $\bar{I}_{\mathbf{D}}$. With \mathbf{t}' given by (23)₃, equations (24) & (25) allow evaluation of σ_x , σ_y and τ as follows:³³

$$\sigma_x = -\sigma + \phi_1 \dot{\varepsilon} + \phi_2 \dot{\gamma}^2, \sigma_y = -\sigma + 2\phi_1 \dot{\varepsilon} - \phi_2 (\dot{\varepsilon}^2 + \dot{\gamma}^2), \tau = \phi_1 \dot{\gamma} - \phi_2 \dot{\gamma} \dot{\varepsilon}. \quad (26)$$

σ and τ are the applied stresses. In the Melbourne experiments (Budd, et al. (2013), [17]), the lateral extension is unconfined; this corresponds to

$$\begin{aligned} \sigma_y = 0 \rightarrow \sigma &= 2\phi_1 \dot{\varepsilon} - \phi_2 \dot{\gamma}^2, \\ \sigma_x &= -\phi_1 \dot{\varepsilon} + \phi_2 (\dot{\varepsilon}^2 - \dot{\gamma}^2). \end{aligned} \quad (27)$$

Moreover, eliminating between these two expressions ϕ_1 and then substituting the result into (26) with $\sigma_y = 0$ allows evaluation of ϕ_2 and then ϕ_1 , (for details see Morland & Staroszczyk (2019), [130], viz.,

$$\phi_1 = \frac{\dot{\gamma} \tau - \dot{\varepsilon} \sigma}{\dot{\gamma}^2 - 2\dot{\varepsilon}^2}, \phi_2 = \frac{2\dot{\varepsilon} \tau - \dot{\gamma} \sigma}{\dot{\gamma}(-2\dot{\varepsilon}^2 + \dot{\gamma}^2)}. \quad (28)$$

³³ In this process a further redundant equation,

$\frac{1}{3}(-2\sigma - \sigma_x - \sigma_y) = -\phi_1 \dot{\varepsilon} + \frac{1}{3} \phi_2 (\dot{\varepsilon}^2 + \dot{\gamma}^2)$, (25) arises and is used. This equation is useful in the derivation of the results (28).

Finally, substituting these results into (27)₂ leads to the equation

$$\dot{\gamma}(\sigma_x - \sigma) = \dot{\varepsilon}\tau. \quad (29)$$

This equation delivers σ_x as a function of the measured quantities.

To summarize: The above results, due to Morland & Staroszczyk (2019), [130], are important for the following reasons:

1. In the experiments the strain-rates $\dot{\varepsilon}$ & $\dot{\gamma}$ and the stresses σ & τ are measured or monitored. These are the quantities arising on the right-hand sides of (28); they can be calculated from the data obtained during the experimental performances. On the other hand, $\phi_{1,2}$ are functions of \bar{I}_D & \bar{III}_D , which, on the basis of (21) and (22), are given by

$$\bar{I}_D = \dot{\varepsilon}^2 + \dot{\gamma}^2, \quad \bar{III}_D = -\dot{\varepsilon}\dot{\gamma}^2,$$

which are equally point-wise known for these experiments.

2. So, provided the experiment can be conducted as intended and described above, we have pointwise knowledge of $\phi_{1,2}[(\dot{\varepsilon}^2 + \dot{\gamma}^2, \cdot), -(\dot{\varepsilon})_j (\dot{\gamma}^2)_j] = (\phi_{1,2})_j, j = 1, 2, \dots, J$. (30)

The parameter $j=1,2,\dots, J$ counts the number of experimental points. Kriging is popular method to determine the mathematical expressions of $\phi_{1,2}$ as functions of

$$[\dot{\varepsilon}^2 + \dot{\gamma}^2, -\dot{\varepsilon}\dot{\gamma}^2].$$

3. Equation (29) is independent of the response functions $\phi_{1,2}$; since $\dot{\varepsilon}$, $\dot{\gamma}$ and σ, τ are measured or monitored, this relation determines σ_x directly in terms of the measured data. On the other hand, once the $\phi_{1,2}$ -functions are determined by optimization from eq. (30), σ_x can be evaluated from (16)₂, using the optimized $\phi_{1,2}$ -functions. If this expression does not ‘reasonably’ reproduce measured $[(\sigma_x)_j, j = (1, \dots, J)]$ -values, then the Reiner-Riwlin constitutive relation is too restrictive as a constitutive equation for glacier ice.
4. Prerequisites of the results of the solution are that $\bar{D}_{11} = 0$, requesting that the normal strain-rates $\dot{\varepsilon}$ in the x_1 -direction are prevented, whereas those in the x_2 -direction balance those in the x_3 -direction (because of volume preserving). Since the normal stress component in the x_3 -direction is compressive, the corresponding x_1 - and x_2 -components of the strain-rates are extensional (but that in the x_1 -direction is prevented ($\dot{\varepsilon}_x = 0$)), so σ_x is a pressure.

It is evident that in principle combined compression-shear tests can be conducted, in which the scalar functions $\phi_{1,2}[\bar{I}_D, \bar{III}_D]$ of a Reiner-Riwlin fluid for ice can be determined. Morland & Staroszczyk (2019), [130], used data from Budd, et al. (2013), [13], and plot the data-related quantities $(-\bar{III}_D)^{1/9}$ against $(\bar{I}_D)^{1/6}$ and obtain the plot in (Figure 11). If the ice specimens in these experiments were insensitive to variations of \bar{III}_D , these experimental points should lie on the $(\bar{I}_D)^{1/6}$ -axis. Some experimental points indeed lie on this axis, but not all. Morland (2007), [113], shows that for independence of the $\phi_{1,2}$ -functions of \bar{III}_D , a unique relation $\bar{III}_D = f(\bar{I}_D)$ must exist, but co-axiality of the strain-rate and stress deviators, viz.,

$$\mathbf{D} = a(T, ?) (II_{t'})^{(n-1)/2} \mathbf{t}' \quad (31)$$

cannot be maintained. So, (Figure 11) supports the Reiner-Riwlin structure of the constitutive relation $\bar{\mathbf{D}} = \bar{\mathbf{D}}(\mathbf{t}')$.

The result also confirms for the Budd, et al. (2013) experiments, [17], that the deviation from co-axiality is not small. In almost all cases ice is treated as volume preserving ($\text{div } \mathbf{v} = 0, I_{t'} = 0$) and the momentum equation is applied in the Stokes approximation, i.e., the acceleration terms are ignored.

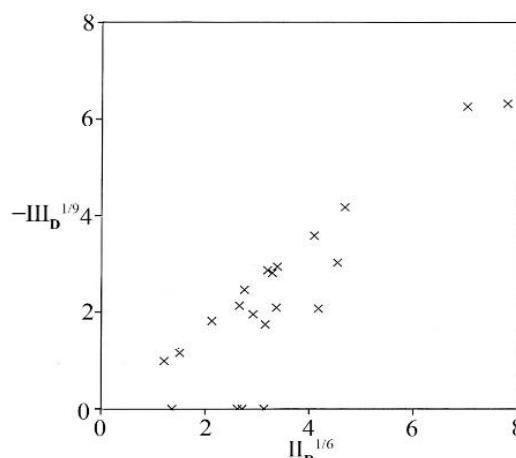


Figure 11: Data points in the invariants plane defined by the axes $(\bar{II}_D)^{1/6}$, $(-\bar{III}_D)^{1/9}$ for 21 experiments of Budd, et al (2013), from Morland & Staroszczyk (2018), [130].

Applications to Climate Dynamics

Field equations for cold and temperate ice regions

Glaciers, ice sheets and ice shelves in climate relevant analyses are mostly treated as polycrystalline isotropic power law fluids, but restricted to constitutive relations of the form ‘ice plus inclusion of water’. This is so, even though careful experiments have shown that a complete Reiner-Riwlin structure with non-coaxiality of the stress-strain rate relation is likely better modeling the creep behavior of isotropic ice. The concentration of the water in the ice varies according to the thermal regime that prevails as a result of input of geothermal heat from the interior of the Earth, the heat flow exchange at the ice-atmosphere interface and the water production per unit mixture volume due to viscous heat of the mixture. Cold ice is ice with vanishing water content and temperature below the freezing point, while temperate ice is at the pressure melting point, in which the temperature is related to the pressure.

Furthermore, the contact surface of the cold and temperate ice – this surface is called the Cold Temperate Transition surface (CTS) at which the Clausius-Clapeyron equation must hold, is an internal singular surface. The thermomechanical conditions that are described by the jump conditions of the field equations in the cold and temperate subsets of the polythermal ice mass determine locally, how the melting and freezing processes evolve along the CTS. On the other hand, within the temperate ice the production of moisture equals the production of water mass per unit volume.

The governing field equations of ice in the cold ice region are the balances of mass, momentum (in the Stokes approximation) and internal energy. The constitutive relations describing the creeping flow are here given by the Reiner-Riwlin strain-rate–stress relation, eqs. (4) & (5) (i.e. not the usual Glen-Steinemann law), and the heat flux vector \mathbf{q} is given by Fourier’s law of heat conduction, whereas the stress evolution is governed by the Stokes equations, and the temperature evolution follows from the internal energy balance. These equations are for a density preserving fluid of the form

$$\begin{aligned} \operatorname{div} \mathbf{v} &= 0, \rho \dot{\varepsilon} = \rho c_p \dot{T}, \\ \rho \dot{\mathbf{v}} &= -\operatorname{grad} p + \operatorname{div} \mathbf{t}' + \rho \mathbf{g} = \mathbf{0}, \mathbf{D} = \psi_1 \mathbf{t}' + \psi_2 [(\mathbf{t}')^2 - \frac{2}{3} \mathbf{II}' \mathbf{1}], \\ \rho \dot{\varepsilon} &= \operatorname{tr}(\mathbf{t}' \mathbf{D}) - \operatorname{div} \mathbf{q}, \mathbf{q} = -\kappa \operatorname{grad} T, \end{aligned} \quad (32)$$

and $\psi_{1,2}$ are given by eqs. (5), while κ is generally assumed to be at most a function of T . In eqs. (32) \mathbf{v} , p and T are the unknown fields.

Correspondingly, in temperate ice regions, the physical laws are a class-I binary mixture; i.e. the equations comprise of the balances of mass, momentum and energy for the mixture as a whole plus a balance for the water mass. Interpreting the equations on the left-hand side of equations (32) as the mass, momentum and energy balances of the mixture as a whole, then these equations must be complemented by the balance of the water mass,

$$\rho \dot{w} = -\operatorname{div} \mathbf{j} + C, \quad (33)$$

in which w is the moisture content per unit mass, \mathbf{j} is the moisture flux and C the moisture production rate per unit volume of the mixture. For this mixture model constitutive relations are needed for \mathbf{j} , C and the $(\mathbf{t}', \mathbf{D})$ -relation, apart from the Fourier law for \mathbf{q} . For \mathbf{j} we postulate Fick’s law

$$\mathbf{j} = -v \operatorname{grad} w, \quad (34)$$

where v is the diffusivity, and $v \equiv 0$ may equally be a possible option. Moreover, considering that all internal energy production of the mixture is instantly used up by melting, energy and mass production can be interrelated with the aid of the latent heat of fusion L per unit volume of the mixture. $\rho \dot{\varepsilon}$ can, thus, be identified with L , the latent heat of fusion, while in the energy equation the heat flux contribution $\operatorname{div} \mathbf{q}$ is ignored, as $\operatorname{grad} T$ is small, since $T = c_{ph} p$, where c_{ph} is the Clausius-Clapeyron constant ($c_{ph} = 0.74 \times 10^{-2} K bar^{-1}$) that is small. So, $CL = \operatorname{tr}(\mathbf{t}' \mathbf{D})$ and

$$\operatorname{tr}(\mathbf{t}' \mathbf{D}) = \operatorname{tr} \left\{ \mathbf{t}' \left(\psi_1 \mathbf{t}' + \psi_2 [(\mathbf{t}')^2 - \frac{2}{3} \mathbf{II}' \mathbf{1}] \right) \right\} = \operatorname{tr} \left\{ \psi_1 \mathbf{t}'^2 + \psi_2 [(\mathbf{t}')^3 - \frac{2}{3} \mathbf{II}' \mathbf{t}'] \right\},$$

in which eq. (4) has been substituted, here adequate for a Reiner-Riwlin fluid. Moreover, employing the Caley-Hamilton theorem for the stress deviator \mathbf{t}' , it can be shown that

$$\mathbf{t}'^3 = \left(\frac{1}{2} I_{(t')^2} \right) \mathbf{t}' + III_{t'} \mathbf{1}.$$

Substituting this into the above formula yields

$$tr(\mathbf{t}'\mathbf{D}) = tr \left\{ \psi_1 \mathbf{t}'^2 + \psi_2 \left[III_{\mathbf{t}'} \mathbf{1} + \left(\frac{2}{3} II_{\mathbf{t}'} + \frac{1}{2} I_{\mathbf{t}'}^2 \right) \mathbf{t}' \right] \right\} = \psi_1 tr[(\mathbf{t}')^2] + 3\psi_2 III_{\mathbf{t}'} \quad (35)$$

It follows that

$$C = \frac{\psi_1 tr[(\mathbf{t}')^2] + 3\psi_2 III_{\mathbf{t}'}}{L} \quad (36)$$

If Nye's co-linearity assumption of \mathbf{D} and \mathbf{t}' is imposed, then $\psi_2 = 0$, and

$$C = \frac{tr(\mathbf{t}'\mathbf{D})}{L} = \frac{\psi_1 tr[(\mathbf{t}')^2]}{L}, \quad (37)$$

as stated e.g. by Hutter (1983), [86]. Furthermore, according to Lliboutry (1979), [113], the presence of melt-water can significantly affect the constitutive relation of the stress deviator. He concludes that the constitutive equation for the creep law remains formally valid but coefficients depend now on the moisture content w and not on T , which is related to the pressure via the Clausius-Clapeyron equation. For the Reiner-Riwlin, fluid this means that

$$\{\psi_{1,2}, v\} = Fcts(II_{\mathbf{t}'}, III_{\mathbf{t}'}, w), II_{\mathbf{t}'} = \frac{1}{2} tr(\mathbf{t}')^2, III_{\mathbf{t}'} = \det(\mathbf{t}') \quad (38)$$

Analyses of the constitutive functions for the creep laws for temperate ice have also been given with more details than here by Blatter & Hutter (1991), [15], Fowler & Larson (1978), (1980), [49, 50], Greve (1995, '97, 2000), [67-70], Hutter (1982, '83), 2019), [83, 86, 90], Hutter, et al. (1988), [88], however only for the situation that the strain-rate is collinear to the stress deviator. For the Reiner-Riwlin fluid, the equation for the moisture content takes the form

$$\rho \dot{w} = v \nabla^2 w + \frac{\psi_1 tr[(\mathbf{t}')^2] + 3\psi_2 III_{\mathbf{t}'}}{L} \quad (39)$$

Note that C depends also on the third stress deviator invariant, if stress and strain-rate are not co-axial. In any concrete situation, the union of these statements defines the initial boundary value problems (IBVP) that must mathematically be solved to arrive at a set of quantifications of the variables as functions of space and time that will provide information on the climate relevant questions for which the IBVPs were formulated.

Phase change properties of a viscous-heat conducting fluid are well known, see e.g. Hutter (1983 or 2019), [86, 90], but need carefully be introduced for materials, which are kinematically handled as density preserving. The difficulty is that there cannot be a pressure melting formula and neither a Clausius-Clapeyron formula in a density preserving material. We, thus, will treat here compressible viscous fluids and present phase change properties of such materials and then will come back to the case of a density preserving material.

A phase change surface of a continuous material is defined as a special singular surface at which the temperature and the velocity component that is tangential to this surface are continuous. Thus,

$$[T] = 0 \text{ and } [v_{||} t_i] = 0 \text{ on a phase change surface } S, \quad (40)$$

where $v_{||}$ is the speed tangential to S , and t_i denotes a unit vector in the direction of $v_{||}$. Apart from (40), also the ordinary jump conditions of the balance laws of mass, momentum and energy hold, as does a jump condition of entropy. These laws are

$$\begin{aligned} & \llbracket \rho(v_i - u_i)n_i \rrbracket = 0, \\ & \llbracket t_{ij}n_j \rrbracket - \llbracket \rho v_i(v_j - u_j)n_j \rrbracket = 0, \\ & \llbracket t_{ij}v_i n_j - q_i n_i \rrbracket - \llbracket \rho(\varepsilon + \frac{v^2}{2})(v_i - u_i)n_i \rrbracket = 0, \\ & \llbracket \frac{q_i n_i}{T} \rrbracket + \llbracket \rho \eta(v_i - u_i)n_i \rrbracket = 0, \end{aligned} \quad (41)$$

in which \mathbf{v} is the velocity vector of a particle, and \mathbf{u} is the velocity vector of the corresponding geometric point on the singular surface S . These equations can be found in any book on continuum mechanics, e.g. Hutter (1983), [86], or Hutter & Wang (2016, 2018), [89]. Incorporating (40) into (41) yields

$$\begin{aligned} & \llbracket \rho(v_i - u_i)n_i \rrbracket = 0, \\ & \llbracket t_{ij}n_j \rrbracket - \llbracket \rho v_i(v_j - u_j)n_j \rrbracket = 0, \\ & \llbracket q_i n_i \rrbracket = \llbracket \frac{1}{\rho} t_{ij} n_i n_j - \varepsilon - \frac{1}{2} (v_k - u_k)(v_k - u_k) \rrbracket \rho(v_i - u_i)n_i, \\ & \quad \frac{1}{T} \llbracket q_i n_i \rrbracket + \llbracket \rho \eta(v_i - u_i)n_i \rrbracket = 0. \end{aligned} \quad (42)$$

Hutter (2019), [90] gives proof of these relations. In this proof, the fact that the components of \mathbf{v} and \mathbf{u} that are tangential to S are continuous, have been employed.

In thermostatic equilibrium one has $q_{i|E} = 0$. It follows, the term on the right-hand side of (42)₃ must vanish. This can be achieved by setting either one of the two factors equal to zero. We set $\mathbf{v} = \mathbf{u}$, because the other alternative would lead to complicated expressions, not well interpretable; in words: the surface of phase change in thermodynamic equilibrium is material.

Next, eliminating $\llbracket q_i n_i \rrbracket$ between (42)₃ & (42)₄ and then using (42)₁ leads to the entropy jump condition

$$\llbracket \eta \rrbracket + \frac{1}{T} \llbracket \frac{t_{ij} n_i n_j}{\rho} \rrbracket - \varepsilon - \frac{1}{2} (v_k - u_k)(v_k - u_k) = 0. \quad (43)$$

When both phases are in thermostatic equilibrium, then

$$\llbracket t_{ij} n_j \rrbracket_{|E} = 0 \rightarrow \llbracket p_{|E} \rrbracket n_i = 0 \rightarrow \llbracket p_{|E} \rrbracket = 0. \quad (44)$$

The pressures on both sides of S equilibrate themselves, so eq. (40) yields

$$\llbracket \eta_{|E} \rrbracket = \frac{1}{T} \llbracket \frac{p_{|E}}{\rho} + \varepsilon_{|E} \rrbracket = \frac{1}{T} \left\{ p_{|E} \llbracket \frac{1}{\rho} \rrbracket_{|E} + \llbracket \varepsilon \rrbracket_{|E} \right\}, \quad (45)$$

an equation that is usually written as

$$\llbracket \mu_{|E} \rrbracket = T \llbracket \eta_{|E} \rrbracket - \llbracket \varepsilon_{|E} \rrbracket - p_{|E} \llbracket \frac{1}{\rho} \rrbracket_{|E} = 0$$

in which $\mu_{|E}$ is called equilibrium chemical potential.³⁴ Hence, in thermodynamic equilibrium, pressure and chemical potential are continuous across the phase change surface S . Another familiar form of (43) is

$$T \llbracket \eta_{|E} \rrbracket = \llbracket \varepsilon \rrbracket_{|E} + p_{|E} \llbracket \frac{1}{\rho} \rrbracket. \quad (45)$$

$T \llbracket \eta_{|E} \rrbracket$ is called the equilibrium latent heat of fusion, $L_{|E}$, implying

$$L_{|E} = \llbracket \varepsilon \rrbracket_{|E} + p_{|E} \llbracket \frac{1}{\rho} \rrbracket.$$

It consists of the jump of the internal energy across S plus the power of working of the stresses due to the specific volume change across S . A different way of writing it is

$$p_{|E} = \frac{T \llbracket \eta_{|E} \rrbracket - \llbracket \varepsilon \rrbracket_{|E}}{\llbracket 1/\rho \rrbracket} = \frac{L_{|E} - \llbracket \varepsilon \rrbracket_{|E}}{\llbracket 1/\rho \rrbracket}. \quad (46)$$

The interpretation of this equation is facilitated, if the relations

$$\llbracket p_{|E} \rrbracket = 0 \& \llbracket \mu_{|E} \rrbracket = 0 \quad (47)$$

are applied. Both, $p_{|E}$ and $\mu_{|E}$ are functions of ρ_I, ρ_{II} and T , i.e. the densities of the two phases and the temperature T which is continuous across S . Therefore, at a given temperature, eqs. (47) may serve as equations to determine ρ_I and ρ_{II} , respectively, as functions of T ³⁵. Once these densities are known, eq. (46) may be used to determine $p_{|E}$ as a function of the temperature alone. Hence, eq. (46) may be written as

$$p_{|E} = \hat{p}_{|E}(T) = \frac{T \llbracket \hat{\eta}_{|E} \rrbracket - \llbracket \hat{\varepsilon}_{|E} \rrbracket}{\llbracket 1/\hat{\rho} \rrbracket}. \quad (48)$$

Differentiating this expression with respect to the temperature yields (see Hutter 1983), [79],

$$\frac{d\hat{p}_{|E}(T)}{dT} = \frac{\llbracket \eta_{|E} \rrbracket}{\llbracket 1/\hat{\rho} \rrbracket} = \dots = \frac{\llbracket \hat{\varepsilon}_{|E} \rrbracket + p_{|E} \llbracket 1/\hat{\rho} \rrbracket}{T \llbracket 1/\hat{\rho} \rrbracket}. \quad (49)$$

The Clausius-Clapeyron equation is the inverse of this expression,

$$\frac{dT}{d\hat{p}_{|E}(T)} = -c = T \frac{\llbracket 1/\hat{\rho} \rrbracket}{\llbracket \hat{\varepsilon}_{|E} \rrbracket + p_{|E} \llbracket 1/\hat{\rho} \rrbracket}. \quad (50)$$

Since the temperature variations in problems involving phase changes are usually small, c in (50) is generally regarded as a constant.

³⁴ This quantity agrees with the free enthalpy of which another denotation is Gibbs free energy.

³⁵ To avoid this numerically rather difficult problem of the determination of sharp interfaces one could perhaps employ here with advantage the phase field theory.

Notice that in a density preserving material $\llbracket \hat{\rho} \rrbracket = 0$ and with $\hat{\rho} \neq 0$, thus $\llbracket 1/\hat{\rho} \rrbracket = 0$; consequently, $c = 0$. The Clausius-Clapeyron equation states in this sharp limit that the melting temperature does not change with the equilibrium pressure. This also means that one must perform the limit to an incompressible continuum in a weak form, i. e. such that the mass balance is approximated by $\operatorname{div} \mathbf{v} = 0$, whilst the Clausius-Clapeyron constant c does not vanish.

In non-equilibrium the chemical potential must be defined by

$$\mu := T\eta + \frac{t_{ij}n_i n_j}{\rho} - \varepsilon - \frac{1}{2}(v_k - u_k)(v_k - u_k), \quad (51)$$

as suggested by (40), and relations (39)_{1,2} and (43) must be replaced by

$$\begin{aligned} \llbracket \rho(v_i - u_i)n_i \rrbracket &= 0, \\ \llbracket t_{ij}n_j \rrbracket - \llbracket \rho v_i(v_j - u_j)n_{ij} \rrbracket &= 0, \quad (52) \\ \llbracket \mu \rrbracket &= 0. \end{aligned}$$

The exact exploitation of (52) has so far not been achieved. This is the reason why one assumes near equilibrium behavior and restricts u_i and v_i to small values. This is tantamount to ignore terms of higher order in u_i and v_i . In this approximation all results are similar to the previous ones. In particular (52)_{2,3} become approximately

$$\llbracket t_{ij}n_j \rrbracket \cong 0, \llbracket \eta \rrbracket + \frac{1}{T} \llbracket \frac{t_{ij}n_i n_j}{\rho} - \varepsilon \rrbracket \cong 0. \quad (53)$$

$$\text{Hence} \quad L := T\llbracket \eta \rrbracket \cong \llbracket \varepsilon \rrbracket - t_{ij}n_i n_j \llbracket \frac{1}{\rho} \rrbracket. \quad (54)$$

Since $t_{ij}n_i n_j$ is the traction component normal to S , we may write $p_N := -t_{ij}n_i n_j$ and now obtain the pressure melting formula as

$$p_N := (L - \llbracket \varepsilon \rrbracket) / \llbracket 1/\rho \rrbracket. \quad (55)$$

Hence, in contrast to the Clausius-Clapeyron equation in thermodynamic equilibrium, it is not the thermodynamic pressure entering formula (55) but the normal traction, p_N , exerted on the surface of phase change S , that is positive as a pressure force and negative in tension. It may not only contain a contribution from the thermodynamic pressure but also a viscous contribution. Kamb (1961), [101] has already obtained this result by methods of classical thermo-statics that is derived here with thermodynamic principles. The above presentation follows Hutter (1983), [86].

Shallow flow approximations and Stokes models

The above analysis for the description of the thermo-mechanical response across a phase change surface addresses this behavior on an interior surface where cold and temperate ice touch each other. Such ice masses are also bounded by other surfaces, e.g. the free surface separating ice from the atmosphere, the interfaces between ice & solid ground and ice & water for ice shelves floating on ocean or lake water, see (Figure 12). In a transition of an ice sheet into an ice shelf, a singular line, called grounding line, separates the grounded sheet from the floating shelf. The case, where an ice sheet ‘sits’ primarily on solid ground, but also has floating portions (on sub-glacial lake(s)), also occurs; Antarctica e.g. is an ice continent with a substantial portion resting on sub-Antarctic lakes.

Each glacier, ice sheet and ice shelf has its own set-up and boundary conditions that together with the field equations define its initial boundary value problem. Two limiting situations are sketched in (Figure 12). An ice sheet (left part) rests on solid ground, and the flow is dominantly horizontal with smaller vertical velocity components. This corresponds to deformations, principally governed by shearing. Length scales of field variations are about 10 times the ice sheet depth or larger and normal stress effects can to first order be ignored in comparison to the shear stresses, τ_{xz}, τ_{yz} . The limiting theory, called the shallow ice approximation (SIA, Hutter 1983, [86]) ignores these higher order effects, which, however, can be accounted for in the so-called second order shallow ice approximation (SOSIA, Baral 1999, [8], Baral. et al. 2001, [9]). An ice shelf (right part of (Figure 12)) floats on ocean or lake water; the flow is dominantly horizontally extending or compressing with large $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$ but small vertical shearing τ_{xz}, τ_{yz} . On length scales larger than the shelf thickness the dominant behavior is reminiscent of the response of a membrane with small bending effects at second order. The limiting perturbation

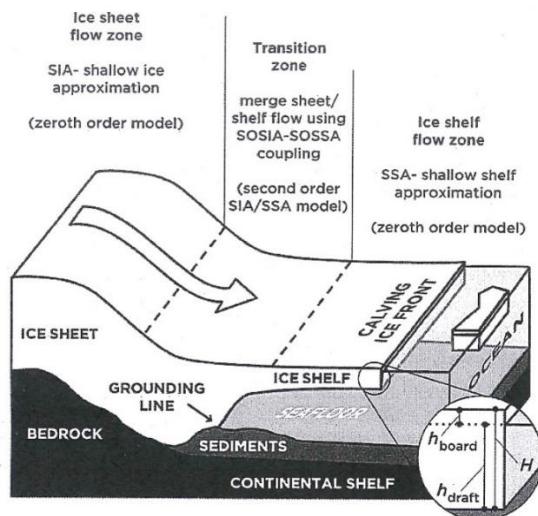


Figure 12 Sketch of a coupled ice sheet--ice shelf system. Upstream (downstream) of the grounding line the ice flow is governed by the SIA-equations (SSA equations). In a transition region across the grounding line, both sheet flow and shelf flow features prevail. So, to match the sheet and shelf flows, the SIA and SSA have to be extended to the SOSIA and SOSSA to properly transfer the flow through the transition zone, from Kirchner, et al. (2011), [103].

theory is here the shallow shelf approximation (SSA, Weis 1999, [176], Weis, (2001), [176], Weis, et al. (1999), [172]) that ignores the vertical shearing. The latter is, however, accounted for by the second order shallow shelf approximation (SOSSA).

The above described properties almost trivially suggest that the transition behavior from an ice sheet to an ice shelf must necessarily be treated by a combination of the SOSIA & SOSSA³⁶, for a graphical sketch, see (Figure 13).

At the outset one ordinarily does not have sufficient information, whether the ice mass is cold or temperate or polythermal.³⁷ More specifically, even if one should have access to the base of such an ice mass and can reasonably estimate where the solid base is cold and where it is temperate, one still has generally no initial knowledge of its thermodynamic state -- cold, temperate or polythermal with unknown location of the CTS. Extensive initial computations by trial and error will have to be conducted just to estimate the initial conditions for a reliable evolution scenario. So far such computations have only been performed by employing the power flow law, even though our review has evidenced that the law (31) requires amendments by replacing the Glen-Steinemann power law by a functional relation reproducing a finite viscosity at zero stress deviator, or even better to use the Reiner-Riwlin fluid law. The standard procedure is to initially assume an ice mass to be wholly cold. This likely corresponds to polar glaciers & ice sheets for which the entire ice mass has temperatures below the melting point except perhaps at, and very close to, the basal surface. If computations should generate small basal-near boundary layers with temperatures above the melting point, then the temperature at these points is set equal to the melting point in this sub-region and the no-slip boundary condition is replaced at the base by a perfect sliding law. A few iterations may suffice to reach convergence.

³⁶ Baral (1999), [8], Baral et al. (2001), [9] Schoof & Hindmarsh (2010), [149], Kirchner et al. (2011), [103]. Weis (1999), [176], Weis et al. (2001), [174].

³⁷ About 50 years ago, Swiss glaciologists said that Swiss glaciers are wholly cold. However, already in the 1980s there were sufficient indications that they safely had to be assumed to be polythermal. An early paper is Haeberli (1976), [75].

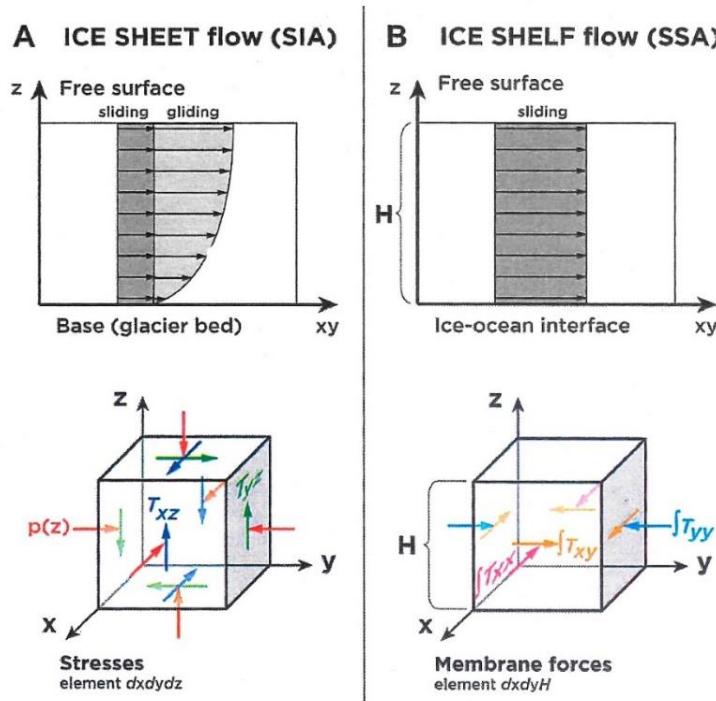


Figure 13: Velocity profiles and stress states for ice-sheets (A) and ice-shelves (B), showing at the top the characteristic vertical velocity profiles that mimic sliding and gliding for SIA and only sliding for SSA; in the bottom row the infinitesimal cubes are shown with the stresses; T_{xx} , T_{yy} , T_{xy} . They are ignored in the SIA, but they exist as membrane forces in the SSA, from Kirchner, et al (2011), [103].

Generally, authors only dispute about a temperature dependent rate factor and restrict their focus on cold ice. They write $a(T, ?)$ as $E(?)a(T)$, in which E is called an enhancement factor and $a(T)$ a ‘standard’ temperature-dependent rate factor, mimicking the monotonic behavior as displayed e.g., in (Figures 9,10). In this process, the appropriate assignment of numerical values to the rate factor $a(T, ?)$ is critical and for a specific concrete functional relation data are not very helpful. They are increasingly spread, as we have shown. Cuffey & Paterson (2010), [32], recommend to employ

$$D = AE_* I I_D^{\frac{n-1}{2}} t', A = A_* \exp \left(-\frac{Q_c}{R} \left(\frac{1}{T_h} - \frac{1}{T_*} \right) \right),$$

$$n = 3, T_* = 263 + 7 \times 10^{-8} p, T_h = T + 7 \times 10^{-8} p, \quad (56)$$

$$Q_c = Q^- = \begin{cases} 6 \times 10^4 & \text{if } T_h < T_s, \\ Q^+ & \text{if } T_h > T_s. \end{cases}$$

Here, T is in degree Kelvin, Q in J mol^{-1} , p in Pa, and R = $8.314 \text{ mol}^{-1} \text{K}^{-1}$; the remaining parameters are $A_* = 3.5 \times 10^{-25} \text{ Pa}^{-3} \text{s}^{-1}$, $Q^+ = 115 \text{ kJ mol}^{-1}$, $E_* \geq 2$ (polar ice under shear).

For details, see Cuffey & Paterson (2010), [32]. However, the arguments in which way this parameterization is justified is not very convincing to us. For instance, the value of the parameter E_* is said to be adjusted according to additional information regarding further implications from other sources, which is far from being specific. This indeterminacy of correct values for A and E_* is not helpful. What transpires is that the compound (AE_*) ought for each ice mass at several locations be determined by careful laboratory tests.

As an example, it is known, that ice that was formed during an Ice Age contains a larger amount of dust than Holocene ice; this affects its deformability. This effect is incorporated by making E_* a function of space & time: $E_*(x, t)$. Ice sheet ices have an approximate value

$$(E_*)_{\text{pleist}} \approx 3(E_*)_{\text{holo}}.$$

Thus, Pleistocene ice is approximately three times softer than Holocene ice³⁸.

The above approach can be used with the Stokes equations³⁹ or the simplified equations of the shallow ice approximation (SIA, Hutter 1983, [79]) or the Parallel Ice Sheet Model (PISM, see Bueler & Brown (2009), [18], that combines the shallow ice

³⁸ A possibility for determining $E_*(x, t)$ is to determine E_* as a function of (x,y)-location and depth using ice core measurements and counting the annual thickness and depth from the free surface to the specimen’s depth.

approximation with the shallow shelf approximation (Weis, et al. 1999), [159]. This model combines the horizontal shear deformation of vast ice sheets and small sliding with the longitudinal straining of ice shelves and ice streams, in which sliding effects near the basal boundary layer may be dominant. According to Greve (1997), [60], for the SIA to be a valid approximation the horizontal length scales of the topography must be at least of the order of 10km. At shorter wave-length resolutions, Stokes models are generally needed and used⁴⁰. In the same spirit, the PISM hybrid model has been used repeatedly to simulate dynamics of ice fields⁴¹.

Schoof (2003), [133] has proposed a proposition for applications of PISM to situations with typical relatively small undulation scales (smaller than ~10 ice depths but larger than 1 ice depth). His scheme introduces a further multiplication factor θ , extending the rate factor in (31) to (θAE_*) with $\theta \in [0, 1]$.⁴² ‘The Schoof scheme assumes that the typical length scale, on which the topography changes are much greater than the ice thickness and much less than the lateral extent of the ice body – an assumption that is not always fulfilled [...] for glaciers and ice-fields. [His] scheme is not valid near nunataks or near ice margins’, (Imhof, et al. (2019), [85], who give a more detailed account of the foundation, validity ranges of the Elmer/Ice and PISM software and the application of these to ice sheets at the LGM).

Geometries for ice sheet extents at the Last Glacial Maximum (LGM, ~24.000 years BP) for the Alpine ice-field have been reconstructed on the basis of positions of terminal moraines and erratic boulders⁴³. Moreover, trimlines were identified with the maximum ice surface elevation at the LGM. Geomorphological reconstructions evidenced for the Alpine ice field details regarding dominant flow domains, flow directions, and ice-free rock formations (nunataks), constrained by bed topography⁴⁴. Imhof, et al. (2019) report: ‘While the reconstructed maximum ice extent can be matched fairly well, the models produce ice thicknesses much greater than suggested by geo-morphological evidence, on average 500 m thicker for the Rhine Glacier (Becker, et al. (2016), [11]), and 800 to 861 m thicker for the Rhone Glacier (Becker, et al. (2017), [12], Seguinot, et al (2018), [152]). Only the modeling study by Cohen, et al. (2018), [29] is able to match the reconstructed ice surface elevation’.

If we understand the theoretical basis of these software applications properly, the authors, who try to numerically reproduce the ice fields at the LGM employ software that does not explicitly predict the evolution of the moisture content in temperate ice regions. Their models nowhere employ or describe the distribution in space and time of the moisture content in the temperate ice domain. The models are, thus, not able to describe the growth or retreat of the evolution of these regions. This important additional complexity will influence the total melting rate of such ice fields that is likely to gain significance in today’s climate scenarios. Moreover, Schoof’s (2003), [149], additional rate factor θ functions as a smoothing operator of the basal sliding resistance, yet the suggested smoothing operation may at best be interpreted as the supposition of perfect sliding at the rough bed that is transferred to a viscous sliding law on the smoothed-out bed. Moisture mass production is not involved.

So, we must treat temperate ice as a mixture of ice with water inclusions⁴⁵ for which the balance laws of mass, momentum and energy are complemented by a balance of mass for the water, formulated as a diffusion equation for this moisture content (defined as the ratio of water mass to mixture mass, Hutter (1982), [83]. The constitutive equation for the mixture stress employs so far the Glen-Steinemann power law whose rate factor depends on the moisture content (that replaces the temperature). The temperature on the other hand stays at the melting point via the Clausius-Clapeyron relation. Moreover, the moisture production rate in the moisture diffusion equation is related to the power of working in the energy equation. The structure of the emerging partial differential equations is parabolic with non-vanishing moisture diffusivity v , else hyperbolic. The temporal evolution of polythermal ice sheets inferred by using the evolution of the class-I mixture of polythermal ice is today (2019) just at its initial steps, see Greve (1997, 2000), [69, 70], Seddik, et al. (2017), [151]. Intensive work on this model in the near future is pressing.

The theoretical parts of the IBVPs have kept mathematically oriented glaciologists and climatologists busy since the 1980s. Yet, to integrate the ‘exact’ Stokes problem computationally is still very difficult, the reason being that integration over a climate cycle (of the order of 100’000 years) or more, to reliably obtain present-day initial conditions for geometry, ice velocity and temperature distributions, is mainly a question of CPU times⁴⁶. To circumvent this, singular perturbation methods were employed. It led to the

³⁹ A prominent software, integrating the Stokes equations by the finite element method, is called ‘Elmer/Ice’, see Gagliardini et al. (2013), [52].

⁴⁰ Leysinger-Vieli & Gudmundsson (2004)[98], Le Meur et al. (2004), [97], Adhiraki & Marshall (2013), [2], Braedstrup et al. (2016), [16], Seddik et al. (2017), [135].

⁴¹ Golledge et al (2012), [59], Yan et al. (2018), [161].

⁴² The value for θ follows from a smoothing procedure of the basal topography, see Schoof (2003), [133]

⁴³ Benz-Meier (2003),[13], Bini et al. (2009), [14], Coutterand (2010), [31],

⁴⁴ Florineth & Schlüchter, (1998), [41], Florineth (1998), [42], Benz-Meier (2003), [13], Kelly et al. (2004), [102], Bini et al. (2009),[14], Coutterand (2010), [31].

⁴⁵ Incidentally, a distribution of the water production could also be estimated with the models not explicitly using a moisture mass balance equation by evaluating a posteriori in the temperate region the moisture production rate C according to eq. (37).

⁴⁶ Numerical solutions of the Stokes equations are today (2019) only possible for realistic territories for integrations over small times into the future, e.g., $n \times 1000, n < 5$ years. However, even under such conditions and computations of a number of climate scenarios the affordable CPU times are excessively large to employ the full Stokes equations, see, however, Seguinot et al.(2018), [152], who achieved 120.000 years of integration.

shallow-ice and shallow-shelf approximations [SIA, SSA]⁴⁷ and their second order improvements [SOSIA, SOSSA]⁴⁸. Today, there exist several open source programs that solve the Stokes formulations for small-scale (in space and time) analyses⁴⁹, for SIA and SSA problems (Greenland, Antarctica, and the Alps)⁵⁰ and Earth-embracing climate studies. To improve this, Schoof's (2003) [149], additional rate factor θ operates as a smoothing operator of the basal sliding resistance, yet the suggested smoothing operation may at best be interpreted as the superposition of perfect sliding at the rough bed that is transferred to a viscous sliding law on the smoothed-out bed. Moisture mass production is not involved. Yet, to adequately model the melting rates and, therefore, the moisture evolution in the likely temperate basal boundary layer correctly, high resolution of the rough basal geometry is unavoidable.

The scientific reports and papers coming out annually are large. They are concerned e.g. with the deglaciation of the mountainous glaciers and ice caps⁵¹, the ice loss of Greenland⁵² and West Antarctica⁵³, coupled with sea level rise (of the order of up to 1m at the end of the year 2100⁵⁴. They affect the global circulation of the ocean currents⁵⁵ and the Earth's climate⁵⁶ and likely leading to millions of displaced people (e.g. of Bangladesh)⁵⁷.

In computations of ice sheet or ice shelf performances over millennia the various domains bounding the evolving ice mass may not be treated as passive rigid continua. The variation of the weight of an ice sheet due to the mass loss/gain by external climate driving requires considering the lithosphere and asthenosphere as heat conducting nonlinearly viscous (plastic) continua. These exert important effects on the non-steady response of the climate variations in time and space on the Earth's surface, and this affects the melting and freezing processes of the ice on the Earth surface. Such effects are incorporated in today's ice sheet models.

To summarize

Today's (2019) state of the art when employing the Stokes equations of the creeping ice flow dynamics suffers from the following deficits:

- Present programs are not flexible enough to apply them for sufficiently long climate scenarios of the order of several (tren to hundred) thousand years.
- This software does not strictly account for the two different thermodynamic states – cold and temperate – that may exist in glaciers and ice sheets with an internal cold-temperate transition surface where the two regions touch.
- The material response is exclusively based on the Glen-Steinemann flow law that has been shown not to be adequately match able with known laboratory experiments. The quadratic terms of the Reiner-Riwlin fluid need to be accounted for, if satisfactory agreement with measured data in multiaxial states of stress is to be obtained.

Summary

We have outlined in the first half of this review that it lasted several centuries until a rational physical understanding of the motion of glaciers (and ice sheets/shelves) had developed. According to the early understanding large ice masses on Earth were thought to be rigid and non-movable, then sliding rigid objects and finally bodies that move like very viscous fluids, similar to honey or a dough. This state of knowledge was achieved by the mid 19 hundred, but it took until the mid-1950s until it was recognized by specialists that this behavior could be categorized as a subject of rheology. It needed the detailed laboratory experiments of Glen and Steinemann to understand the creep of cold ice as a substance whose deformation could be subjected to a mathematical model, known as a nonlinear heat conducting fluid with power law rheology. We could have stopped our review at this point and could have reported how most glaciological specialists spent their time by perfecting the numerical-mathematical approaches for a great number of realistic glaciers, ice sheets and ice shelves on the globe; this would probably have made an acceptable review report. The glaciological community would likely have been happy with this, as the model would stay in conformity with Nye's

⁴⁷ Fowler (1977), [48], and Weis, Greve & Hutter (1999), [176], and Calov, (1994), [20], Calow & Hutter (1996), [21] and (1997), [22].

⁴⁸ Baral (1999), [8], Baral et al. (2001), [9], Schoof & Hindmarsh (2010), [149], Kirchner et al. (2011), [103], Weis (2001), [174], Weis, et al. (1999), [176].

⁴⁹ Approximately 1000m element lengths or somewhat less.

⁵⁰ Becker et al. (2016), (2017), [11,12], Calov (1994), [20], Calov & Hutter (1996), [21], (1997), [22], Greve & Hutter (1995), [68].

⁵¹ e.g. Jouvet et al. (2009, 2011), [99,100], Adalgeirsdóttir et al. (2011), [1], Zekollari et al. (2014, 2017), [179,180], Réveillet et al. (2015), [144], Gilbert et al. (2017), [54].

⁵² e.g. Fürst et al. (2015), [51], Vizcaino et al. (2015), [169], Goelzer et al. (2017), [66], Calov et al. (2018), [23], Rückamp et al. (2018), [146].

⁵³ Favier et al. (2014), [39], Cornford et al. (2015), [30], Feldmann and Levermann (2015), [40], Golledge et al. (2015), [39], Winkelmann et al. (2015), [177], Pattyn et al. (2017), [140].

⁵⁴ De Conto and Pollard (2016), [35].

⁵⁵ e.g. Swingedouw et al. (2008), [160], Gierz et al. (2015), [53], Becker et al. (2016), [11], Jackson and Wood (2018), [98].

⁵⁶ e.g. Vizcaino et al. (2008, 2010), [167, 168], Lenaerts et al. (2015), [111].

⁵⁷ Neumann et al. (2015), [118], Davis et al. (2018), [33].

(1956), assertion [135], (i) that the third invariant does not affect the strain-rate-stress relation and (ii) strain rate and stress deviators are coaxial. We preferred to continue by searching for possible pitfalls that may be contained in this model. However, already in 1980, a committee of ice researchers alerted in a review of experimental studies on the non-negligible effect of the third invariant – a call that has largely been ignored (see Hooke & Mellor and 11 others (1980)).

Already in the late 1950s Steinemann conducted combined shear-compression tests. Glen took these data and checked whether they would be in conformity with the co-axiality postulate. His results are summarized in (Figure 8). If Nye's conjecture would be correct, all theoretical curves plotting Π_D against $\Pi_{t'}$ should collapse to single curve. This is not so, and the experimental points reasonably follow the theoretical curves, clearly separated. It follows the co-axiality of \mathbf{D} and \mathbf{t}' is not supported by experiments. These facts are known since 60 years. We do no longer have to analyze the situation that the co-axiality holds but dependence on the second and third invariants may match with pressure-shear experiments. It is simply mandatory to employ for the nonlinear creep law of polycrystalline ice the full Reiner-Riwlin constitutive relation. Morland and associates have alerted to all this already almost 20 years ago.

A further weakness of the power law parameterization of polycrystalline isotropic ice is that a polynomial fit of degree 3 of the experimental curves do better fit the individual experimental data (Smith & Morland). However, a better fit of data from different authors or sites cannot be obtained this way.

In ice sheet computations of ice shields e.g. for the Alps (and other large ice masses) at the LGM, the basic concepts illustrated above have restricted attention to power laws of cold ice with adjustments of the rate factor in a rather ad-hoc manner to local geomorphologically inferred ice sheet geometry.

Finally, we have not found any substantial application of the mathematical theory of polythermal ice masses. The application of their mathematical theory would need to use the class-I mixture theory, in which the moisture (mass) balance equation would be employed in the temperate ice region, and in which the moisture jump condition would be employed at the Cold-Temperate Transition Surface (CTS). This would give a better knowledge of the water budget in such ice fields.

Furthermore, whereas the mathematical model for polythermal ice masses has been proposed more than 30 years ago (Fowler 1977, [48], Hutter 1982, [83] with improvements by Greve 1999, [7]), its application to realistic situations still awaits its explicit use.

In conclusion, this review has disclosed two extensions of present-day ice sheet models, (i) the replacement of the power law rheology by a full Reiner-Riwlin parameterization and (ii), the explicit use of the Class-I mixture model to properly account for the phase change processes in the temperate ice regions and along the CTS.

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